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Technical Report 32-1438

Strain-Energy Size Effect

J. Glucklich

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Preface

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Abstract

The effect of the physical size of a specimen upon the ***initiation*** of fracture of materials is in accordance with statistics of flaw distribution. The effect of size upon *total* fracture is as above, plus the effect during stable crack propagation. Stability of cracking can be because of (1) energy dissipation, (2) load relaxation, or (3) crack orientation. Only (1) reflects a material property. The *energy-dissipation stability* is affected by the strain-energy content (and therefore by size) in such a way that the higher the energy, the earlier this stability transforms to instability. Consequently, the larger the specimen, the lower the breaking stress and the ductility that accompanies the cracking. A possible explanation is presented in terms of dynamic effects caused by an excess in the energy released over the energy absorbed. These dynamic effects influence the stability of the propagating crack in a manner in which the size of the specimen plays a dominant part.

The behavior of three broad groups of materials is examined from the viewpoint of crack stability. These are *ductile* (mainly soft metals), *semiductile* (materials such as concrete and gypsum), and *brittle* (glass). The conditions favoring instability are listed, and the various materials are classified in accordance with their position relative to a transition size. Examples of the effects of size in various materials are cited, and it is shown that existing theories are unable to explain all of the observations, either qualitatively or quantitatively. The proposed theory of a strain-energy size effect seems to fill these gaps satisfactorily.

It is speculated that, in general, every material has two constants that fully describe its resistance to fracture: γ' and G_i. These involve the critical strain-energy release rate G_c. Here, γ' is the limiting value of $\frac{1}{2}$ G_c when size increases to infinity; G_i is the limiting value of G_c when size decreases to zero. In practice, γ' controls the initiation of cracking and G_c (not G_i) controls the onset of instability. Whereas γ' is independent of specimen size, a study should be made of the size dependence of G_c. Evidently γ' is also the true design criterion for very large, ductile members, and G_i is the design criterion for very small, brittle elements.

Transition-size curves are proposed (in analogy to transition–temperature curves), and the positions of the transition for some materials are roughly indicated.

Strain-Energy Size Effect

I. Introduction

Several phenomena encountered by investigators in the field of strength and fracture of materials cannot be explained by recognized laws. For example, the brittle behavior of large-scale, mild-steel elements; the plasticity recently shown to exist in glass; and the marked size sensitivity of fatigue specimens all are little-understood phenomena. An attempt will be made to show that these and other unexplained observations are actually manifestations of a single law, which is now stated (in a qualitative form at present) under the name of the *strain-energy size effect*.

Size sensitivity of materials is customarily attributed to statistical considerations, and the weakest-link theory is the most widely recognized theory among them. However, it must be realized that such theories are valid only for the *initiation* of fracture; as for *total* fracture, these theories are valid only in those cases where initiation and termination coincide. Such cases are rare because, in most instances, a stage of fracture propagation intervenes between the two events. It is conceivable that, during this intervening stage as well, size will influence strength. The total effect will, therefore, be the sum of the two separate effects on initiation and on propagation.

In this report, attention is focused upon the stage of fracture propagation, which is mainly examined from the standpoint of stability or instability and how these are influenced by the size of the medium. There is evidence to the effect that fracture propagation is a process of very unstable equilibrium so that instability (i.e., fracture) can be easily induced by momentary overloads. It is easy to see how the strain-energy content of the system, and therefore the size of the specimen, may contribute to such overloads. The effect may be crudely demonstrated by a simple experiment.

A sheet of paper is torn into two pieces by first making a short tear to serve as a tear nucleus. Then the paper is pulled perpendicular to the tear in two ways: (1) by pulling from positions very close to the tear and (2) by pulling at the edges of the sheet. Method (1) will yield a slow and stable tearing (cracking); method (2) will at first yield a very limited, slow-growth tearing (cracking), that will almost immediately become fast and uncontrollable. The difference between the two methods lies in the boundary condition around the crack—controlled strain rate in case (1) vs controlled stress rate in case (2). In other words, the difference is in the availability of strain energy, which is limited in case (1) and unlimited in case (2). In case (1), the tearing stress easily adjusts itself to resistance fluctuations offered by the paper fibers; in case (2), when this resistance momentarily drops, an overload develops that leads to instability.

In practice, case (1) is almost nonexistent, as enough strain energy is always available for the creation of overloads. The effect of these in causing instability is not clear. However, some speculations, whose validity has not been checked by experiment, will be made in this report.

The main object of this investigation is to reexamine size-effect observations made of various types of materials in the light of the proposed theory. It is shown that the recognized theories are not sufficient to explain all observations, and that the new theory fills the gaps. The insufficiency of existing theories is not merely a quantitative one. They are qualitatively wrong as well; for example, in their inability to account for the influence of size on ductility.

The organization of this report is as follows: In Sections I–IV, the proposed theory for the causes and mechanism of the strain-energy size effect is presented. (Section V, dealing with the possible mechanisms by which the dynamic effects lead to instability, is highly speculative!) Sections VI–IX present experimental results for three typical groups of materials. In Sections X and XI, the theory is further expanded, and a transition size is proposed.

To prevent ambiguity, some clarification of terminology is necessary. In this report, *crack nucleation* is the formation of crack nucleus *prior* to the application of load. *Crack initiation* is the beginning of growth of the existing nucleus as a result of the application of load. The "Griffith condition" is a term used by some authors to represent the crack-initiation condition defined above. Others use it to describe the instability condition; i.e., the transition from stable to unstable cracking. This inconsistency stems from the fact that, in Griffith's experiments with glass, initiation and instability coincided. In the general case, however, where these are two separate events, separate terms are necessary, and *initiation* and *instability* are recommended. To prevent confusion, the term "Griffith condition" will not be used again in this report.

II. Effect of Size on Fracture Initiation

If a single crack nucleus is present in the specimen, it can be shown that the specimen size will have no effect upon the stress at which this nucleus will start to grow. If U is the elastic energy per unit volume, σ is the nominal stress field, E is Young's modulus, ν is Poisson's ratio, L is a dimension of the specimen, and c is the crack half-axis, then

$$U = U(\sigma, E, \nu, c, L) \tag{1}$$

For his particular case, Griffith (Ref. 1) has shown

$$U = -\frac{\pi c^2 \sigma^2}{E} + f(\sigma, E, \nu, L) \tag{2}$$

The energy criterion of instability states that

$$-\frac{\partial U}{\partial c} \geq \frac{\partial W}{\partial c} \tag{3}$$

where W is the energy absorbed in the formation of a unit surface. Accordingly, the crack will start to grow when

$$-\frac{2\pi c \sigma^2}{E} = \frac{\partial W}{\partial c} = G \tag{4}$$

that is, the specimen dimensions have disappeared in the differentiation, and thus do not affect the occurrence of this event.

Irwin (Ref. 2) approached the problem by considering the stress field in the vicinity of the crack only; from this he derived the energy required to close a small portion α of the crack near its root. This energy is released when the crack opens, and its derivative with respect to α will be the driving force G. Irwin's method enables the derivation of G for various geometries of cracks in terms of the stress-concentration parameters and the nominal stress field. Thus, in all cases, only the conditions in the immediate vicinity of the crack determine its stability, and the specimen size does not enter into consideration. If a population of crack nuclei of a certain density exists, the specimen size will affect the stress at which the first crack will start to grow. This occurs because the severity of the weakest crack will depend upon the number of cracks in the population; i.e., upon the specimen size. Various statistical theories have been developed in accordance with the various crack-size distribution functions assumed by the investigators. These include the Gaussian theory (Ref. 3), the Laplace theory (Refs. 4 and 5), and others.

In summary, size has no effect upon any single crack nucleus, but has an effect upon a population of cracks. This effect is, however, limited to the *initiation* of fracture, and does not include *total* fracture, as erroneously assumed by some authors.

III. Mechanisms of Stable Crack Propagation

For *total* fracture, the effect of specimen size during the stage of stable crack propagation should be examined. Before attempting this, however, the causes of stable crack propagation should be understood.

In the Griffith case (see Ref. 1), no such stability was possible for the following reasons: (1) Griffith assumed an ideally brittle material with only one energy-dissipating mechanism—that of surface tension γ. Furthermore, he assumed γ to be constant. (2) By assuming an infinite body, he avoided the possibility of the relaxation of the applied stress caused by the conversion of energy. (3) He also avoided the possibility of low energy release relative to energy demand by dealing strictly with a case of pure tension. As a result of these three provisions, the energy absorption was, in his case, proportional to the crack length ($W = 4c\gamma$), whereas the energy release was proportional to its square ($U = -\pi\sigma^2c^2/E$). This assured that, beyond the point at which $-\partial U/\partial c = \partial W/\partial c$, the inequality $-\partial U/\partial c > \partial W/\partial c$ persisted; that is, instability was unavoidable.

In practice, the above conditions are not met in most cases; therefore, stability of crack propagation *is* possible. In accordance with the above three conditions, three types of stability are possible, as described below.

A. Energy-Dissipation Stability

Energy-dissipation stability is encountered when the energy absorption is not restricted to surface tension, and, moreover, when the energy absorbed per unit crack length is an increasing function of the crack length; i.e., $\partial^2W/\partial c^2 > 0$. In such a case, when the condition $-\partial U/\partial c = \partial W/\partial c$ is first satisfied propagation is indeed initiated. Because $\partial W/\partial c$ is constantly increasing, however, it is necessary to increase U and $\partial U/\partial c$ by raising σ to maintain the above equality. As a result, the condition

$$-\frac{\partial U}{\partial c} \leq \frac{\partial W}{\partial c} \tag{5}$$

is always satisfied, and stability is assured. Energy-dissipation stability will be discussed further toward the end of this section.

B. Load-Relaxation Stability

Load-relaxation stability is encountered when the load relaxes through limitation of the energy supply.

Assuming the Griffith case in an energetically conservative system, where c_1 is the initial crack length (assumed to be small) and A is the area of the plate, it can be shown from $U = (\sigma^2/2E)A$ that the relaxing stress is

$$\sigma = \left\{\frac{2E}{A}\left[\frac{\gamma A}{\pi c_1} - 4\gamma(c - c_1)\right]\right\}^{1/2} \tag{6}$$

The necessary stress is

$$\sigma = \left(\frac{2E\gamma}{\pi c}\right)^{1/2} \tag{7}$$

Equating Eqs. (6) and (7) to determine the extent of spontaneous propagation, a quadratic equation is obtained, with solutions

$$c = c_1$$
$$c = \frac{A}{4\pi c_1} \tag{8}$$

The crack is thus stabilized through limitation of energy at a distance proportional to the volume of the specimen and inversely proportional to the initial crack length. Paradoxically, as far as spontaneousness is concerned, an initial small crack is, therefore, more dangerous than a large one. Volume is, of course, always conducive to spontaneousness. From Eq. (8) it is clear that, if $c_1 \geq (A/4\pi)^{1/2}$, no amount of spontaneous growth is possible. Actually, this expression is inaccurate because, in evaluating Eq. (8), c_1 was assumed to be small. It is certain, however, that a critical size c_1 exists above which no spontaneous growth will occur. This explains the initial stability of cracks developing at the roots of *deep* notches of brittle materials.

Relaxation of stress through energy limitation can also occur together with type (1) stability (energy-dissipation stability). Thus, at each step of stable propagation, the stress relaxes slightly, and must be re-elevated to allow propagation to continue.

C. Crack-Orientation Stability

When the crack propagates parallel to the stress field (as in uniaxial compression), the amount of energy absorbed remains proportional to its length, whereas the energy released also becomes proportional to the length instead of to the square of the length (as is the case in tension).

If a compressive stress is applied parallel to an existing hairline crack, the crack will not propagate because no energy would be released if it did. However, if the crack has width or, alternatively, if it is at an angle with respect to the field, it will propagate parallel to the compression because some energy will be released. But this release will be small compared to that of a crack perpendicular to the field. The two cases are shown diagrammatically in Fig. 1, where $2c$ is always the total extent of the crack. In tension, it is known that

$$\Delta U = -\frac{\pi c^2 \sigma^2}{E} = -\pi c 2c \left(\frac{\sigma^2}{2E}\right) \tag{9}$$

that is, it may be assumed that the elliptic area enclosed by the dashed line is free of stress, and that the stress in the surrounding area is undisturbed. By analogy, it may be assumed (with a possible small factor of error) that the area free of stress, in the case of compression, is that bounded by an ellipse containing the original flaw plus its extensions. In this case, therefore,

$$\Delta U = -\pi bc \left(\frac{\sigma^2}{2E}\right) = -\frac{\pi b c \sigma^2}{2E} \tag{10}$$

where $2b$ is the front presented by the flaw to the compressive stress.

The obvious difference between the two cases is as follows: in tension, ΔU is proportional to c^2 because of a high disturbance to the field; in compression, ΔU is proportional to c because of the low disturbance. Proceeding as in tension,

$$-\frac{\partial U}{\partial c} = \frac{\pi b \sigma^2}{2E} = \frac{\partial W}{\partial c} = 4\gamma$$
$$\sigma = \left(\frac{8E\gamma}{\pi b}\right)^{1/2} = \text{const} \tag{11}$$

Therefore, in contrast to tension (where σ is inversely proportional to $c^{1/2}$, thus causing instability), in compression, σ is independent of c, and stability can prevail. This type of stability may be the reason for the difference between tensile and compressive strengths of brittle materials. If compressive cracks are stable, as just explained, premature fracture is avoided, thereby permitting the material to reach its mean strength by the gradual transfer of the fracture process from weaker to stronger flaws. In tension, by contrast, the strength is that of the weakest flaw. The problem may, therefore, reduce to that of the difference in strength between the weakest and the mean in the distribution of flaws. Scatter of results in these two types of strength (higher in tension than in compression) supports this view.

Examples of the three stability types are abundant. Irwin (see Ref. 2) and Orowan (Ref. 6) have shown that stable crack propagation in metals is explained by the fact that energy is dissipated by plastic strains ahead of the crack, and that the zone of this plasticity increases with crack length. This makes the energy demand per unit crack area increase faster than the energy release, thereby ensuring stability. Glucklich (Ref. 7) explained, in an analogous manner, the stability of cracking in concrete, with microcracking ahead of the major crack taking the place of the plasticity in metals. Load-relaxation stability is demonstrated in cases where γ is measured by such methods as the lengthwise splitting of strips (Benbow and Roesler, Ref. 8). In such cases, the propagation of the crack unloads the system, and stability is possible. Crack-orientation stability was demonstrated by the conical indentation cracks employed by Roesler (Ref. 9) for measuring γ values of glass. The cracks propagated parallel to the compressive-stress trajectories; therefore, they were stable.

In all three types of stability mentioned above, the catastrophic process is prevented by the lagging of the energy release behind the energy demand. However, only the first type is of a fundamental nature, reflecting a material property; the other two types result from

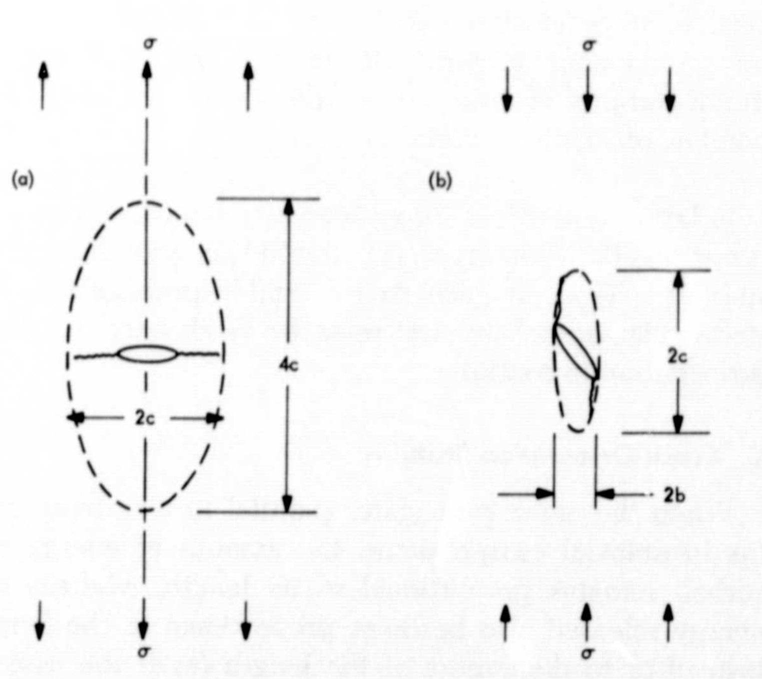

Fig. 1. Energy release: (a) in tension; (b) in compression

geometrical reasons. The capacity of a material to dissipate energy is an intrinsic property contributing to its strength and toughness, whereas the geometrical causes are not likely to affect behavior from the viewpoint of strength and ductility. One simple reason is that if, for example, relaxation of load occurs as described, reloading will restore the original position without altering the ultimate results. By contrast, energy-dissipation stability permits the mobilization of further resistance present in the material. This resistance would not be utilized if premature instability occurred.

It is the premise of this work that energy-dissipation stability exists in every real material, the differences between materials being differences only of degree. Thus, in mild steel and other soft metals—i.e., materials with great capacity for plastic yielding—it is very pronounced. In materials such as concrete, rock, coal, porcelain, etc., energy-dissipation stability is less pronounced. In an atomic sense, no plastic yielding is possible in such materials, but above a certain stress microcracking or crazing takes its place to permit them to manifest macroscopic "plasticity." At the end of this scale are such materials as glass or some glassy polymers, within which it has always been assumed that no kind of plasticity—and hence no energy-dissipation stability—is possible. It is now suggested that even glass has this stability, although it is not exhibited under normal circumstances. As is shown below, however, it may be demonstrated under special conditions.

The behavior of a real material (defined as one showing energy-dissipation stability) vs that of an ideal Griffith material (i.e., one incapable of absorbing energy other than that of surface tension) is shown schematically in Fig. 2, which is taken from Ref. 10. The important difference is the shape of the energy-absorption curve W. This is a straight line in the Griffith material, and an upwards concave curve in the real material. The behavior shown is for a flexible system (or for a large-sized specimen) where no stress relaxation occurs. In a rigid system, at each step the stress will relax somewhat, and will have to be re-elevated before the crack can be extended farther. This, however, will have no effect upon the attained values of σ_c, c_c, and G_c.

IV. Causes of Dynamic Effects During Stable Propagation

The effects of specimen size during stable crack propagation are connected with certain dynamic effects that accompany this stage. It should be noted that, in spite of the use of the term *stability*, the equilibrium between the driving force (i.e., the strain-energy release rate G) and the restraining force (i.e., the energy-absorption rate R) is very delicate (see Fig. 2). Propagation began only after G had become equal to R (or $\partial U/\partial c = -\partial W/\partial c$), and it was soon halted as G dropped very slightly behind the increasing R, only to be repropagated by the further slight increase of load. Thus, the equilibrium depends to a vital extent upon the capability of the loading arrangement to increase the load in unison with the increasing resistance. Any overload may immediately result in instability.

An overload is inherent in the process of fracture regardless of the level upon which it is examined. On the atomic level, it is caused by the fact that the resultant interatomic force has a point of maximum attraction. When the distance between two atoms reaches this critical value, the force required for further extension drops rapidly. Because these two atoms are surrounded by a field in which other atoms are positioned at less than the critical distance apart (i.e., still in the elastic range), this field maintains a constant force between the two atoms in question. Beyond that point, the active force thus exceeds the reaction, the two atoms accelerate away from each other, and kinetic energy is generated. This energy may travel in the form of a wave across the material to add to the potential energy of the pair of atoms next in nearness to the peak, causing the separation of the second pair of atoms. This ideal mechanism, based upon the concept of statistical fluctuations of bond energies, may start a chain reaction that, in most cases, will spread through neighboring atoms. It is known as the "single-bond catastrophe."

On the microscopic level, overloads result when fracture develops through inhomogeneities (such as grain boundaries in polycrystals) or through voids in materials such as concrete. When a void is encountered—i.e., an abrupt drop in energy demand—a sudden excess of energy is available for release.

On the macroscopic level, fillers of various types and shapes (such as stone aggregates in concrete or fibers in polymeric composites) may cause a buildup of stress. These immediately become sources of overload as the fracture advances past them. Examples of such mechanisms, along with the dynamic effects they cause, are presented in Ref. 10.

Most important of the causes of dynamic effects is the case in which the stress condition of fracture is satisfied

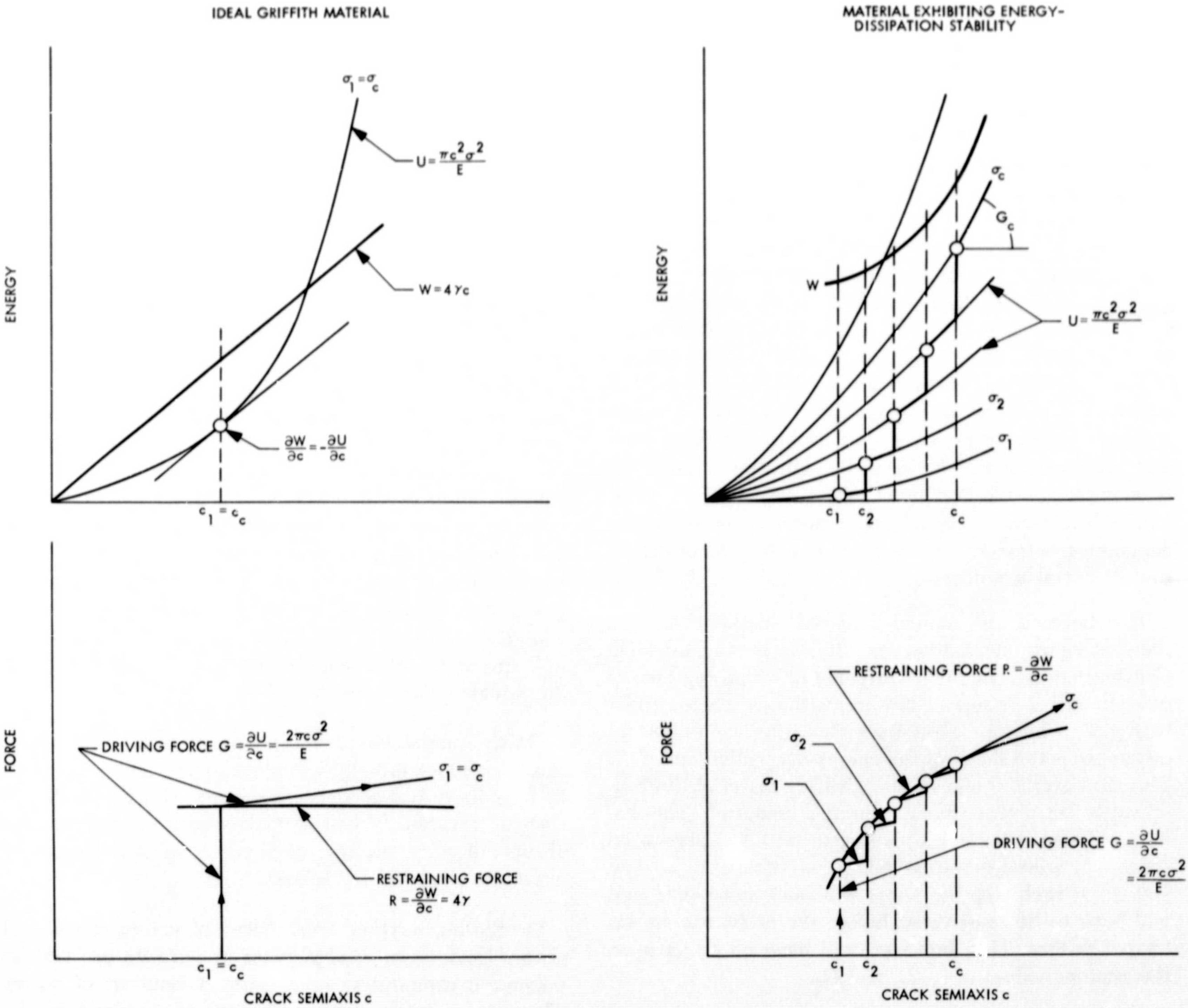

Fig. 2. Energy and force balance for a propagating crack in a flexible—tensile loading system (lower curves are derivatives of upper curves—after Glucklich and Cohen, Ref. 10)

before the thermodynamic condition is satisfied. It is known that, for a crack to propagate, the stress ahead of it must be sufficient to break the existing bonds, while the minimum-energy principle should simultaneously be satisfied. Using atomistic models, Orowan (Ref. 11) and Elliot (Ref. 12) have shown that, when the latter condition is satisfied, the former is also fulfilled. The opposite, however, is not always true. For example, a short and very sharp crack in glass[1] may have a stress concentration sufficient to overcome bonds, but not enough energy release to start propagation. In such a case, the system must be further loaded until the energy balance is favorable. At this time, however, the stress ahead of the crack is far in excess of that necessary for atomic separation, and acceleration will occur.

V. Size Effects During Stable Propagation

It has been shown that stable cracking depends upon a delicate balance between the driving and restraining forces. It has also been pointed out that dynamic effects are inherent in almost all types of fracture. In fact, it can be rationalized that dynamic effects will fail to occur in only two cases: (1) when the forces act directly upon the crack boundaries without the intermediacy of the surrounding material (i.e., an ideally rigid system), so that the driving force changes immediately with the restraining force; or (2) when the crack starts to propagate at a zero stress, so that no energy had accumulated. An attempt will now be made to show that these dynamic effects influence the stability of the propagating crack in a manner in which the size of the specimen plays a dominant part.

Two mechanisms may be triggered by the momentary loss of equilibrium that constitutes the dynamic effect. If a single isolated occurrence, it may generate a compression pulse that will travel to the boundaries and be reflected as a tension pulse. If it happens sequentially at a certain frequency, the entire system may be excited to vibrations. Either the tension pulse or the vibrations may cause the crack to become unstable and the magnitude of both is influenced by the size of the specimen. These two possible mechanisms will now be considered in some detail.

Elastic stress waves have so far been considered mainly in connection with the fast-propagation stage where, it has been shown, their velocity controls the ultimate crack velocity in a manner as yet not fully understood. However, it has also been demonstrated (Refs. 13–16) that, in the slow stage as well, the discontinuous growth of a crack is responsible for the emission of such waves.

Because the material around the crack is in tension, the unbalanced, sudden tearing of a bond will induce an unloading pulse (the potential energy having been changed to kinetic energy) to travel to and return from a boundary as a tension pulse. If this returning pulse is of a sufficient intensity, it may supply the next pair of atoms in the path of the crack with the additional energy needed to make them surmount the energy barrier and separate. If it is of a still higher intensity, the pulse may extend the crack to its critical size, and complete instability will then occur.

Specimen size enters into these considerations in determining the distance through which the pulse travels and, therefore, its attenuation. Its effect is thus opposite to that observed. Exceptions occur in cases of favorable geometrical configurations; e.g., when returning pulses from different boundaries converge upon the critical zone, and reinforce one another to create a momentary high-tension field sufficient to bring about instability. Obviously, such an event is highly unpredictable, and no general law of size effect can be based upon it. It is probable that the many contradictions to the general size rule encountered with some materials (see Sections VI–IX) are partly due to this effect. Gerberich and Hartbower (see Ref. 16), who discovered a unique relationship between the sum of the stress-wave amplitudes and crack growth, found no effect of specimen size upon these amplitudes. The effect of size and geometry in the case of stress waves is thus limited to attenuation and pulse superposition, as described above.

The effect of specimen size in the case of the vibration mechanism may be the main cause of the phenomenon. Such forced vibrations of the entire system may be set up by the repetition of bond rupture occurring during slow crack propagation. Because the source of the disturbance is a point, the vibrations (dilatational and distortional) spread out radially into the surrounding material. Because of its complex geometry, several degrees of freedom, and various modes, the system has a complex dynamic-response behavior. Generally, however, $\omega_n = (K/M)^{1/2}$, where ω_n is the natural frequency, K is the spring constant in a radial direction, and M is the mass. An increase in the size of the specimen without a change of shape will

[1]Griffith estimated inherent cracks in glass to be of the order of 10^{-4} in. Their radius of curvature approaches atomic dimensions.

reduce ω_n because K will decrease and M will increase. The complex dynamic response (i.e., the amplitude vs frequency curve) will simply be shifted to lower frequencies. An increase in size with a change of geometry will cause both a shift to lower frequency and a change of the response curve.

On the other hand, the frequency of excitation ω_e is proportional to the velocity of crack progress. Schardin (Ref. 17) and others have shown experimentally that, during the stage in question, this velocity (starting from zero) always increases, as it depends upon the increasing load, until instability sets in when the velocity of crack progress approaches half the speed of the transverse wave. It thus seems plausible to assume that the larger the specimen, the earlier will ω_e be equal to ω_n (or to a multiple of ω_n for higher modes), at which time the forced vibrations will be resonated and cause instability. The analytical determination of the dynamic response (i.e., amplitude vs frequency) of a real system is an almost impossible task; therefore, it should be determined experimentally. The excitation frequency ω_e can be measured by means of instruments such as piezoelectric transducers or accelerometers (see Ref. 16), using varying sensitivities to register either coarse growth increments (as in concrete) or very fine growth increments (as in glass). At the onset of instability, ω_e should then be compared with ω_n of the system to check the theory.

The above line of reasoning may serve to explain the continuous increase of crack velocity in all of the propagation stages. At a certain initial velocity, ω_e matches with some peak (at low frequency) of the system-response curve. This causes partial resonance, and hence some acceleration of the crack. This acceleration, in turn, increases ω_e, which causes a resonance with a second peak at a higher frequency, further accelerating the crack. Thus, a progressive process develops during which the crack velocity continuously increases and the entire spectrum of natural frequencies is scanned. According to this line of reasoning, the onset of instability occurs when the amplitude gain is sufficient to overcome all remaining resistance; a drastic change in velocity then takes place, but the velocity changes both before and after this event.

The behavior of three broad groups of materials will now be examined from the viewpoint of crack stability. For the present purpose, these are defined as *ductile*, *semiductile*, and *brittle* (ductility meaning here the capacity to dissipate potential energy *during* cracking). For convenience, the conditions that favor instability are listed as follows:

(1) Capacity to dissipate potential energy as small as possible.

(2) Inclusions and voids present at all levels of aggregation and coarseness of grains, all of these being conducive to dynamic effects.

(3) Extremely thin and short cracks preexistent, so that the stress condition favorable to growth is satisfied before the energetic condition.

(4) Small capacity for damping vibrations (i.e., kinetic energy).

(5) Large size (relative to other conditions).

A good representative of the ductile group is mild steel, which does not satisfy condition (1), satisfies condition (2) only partly and only on the microscopic level (grain boundaries), and does not satisfy conditions (3) and (4). Therefore, mild steel is highly stable; only very large specimens or structure members may cause early instability (i.e., may induce brittleness). The addition of alloying elements to steel affects conditions (1) and (2)—the capacity for plastic strains is reduced, and the presence of impurities enhances dynamic effects. Therefore, smaller specimens of these alloys are needed to cause a laboratory-size specimens.

Concrete will serve as an example of the semiductile group. Condition (1) is partly satisfied because concrete—although it has some energy-dissipating capacity (in the form of microcracks, but not real plasticity)—is far behind the real plastic materials in this respect. Condition (2) is completely satisfied on all levels of aggregation. Conditions (3) and (4) are not satisfied. The size necessary to cause transitions from ductility to brittleness is, therefore, medium; in fact, as shown in Section VIII, it occurs with laboratory-size specimens.

Glass is, of course, the prototype of the brittle group. Conditions (1) and (3) are completely satisfied. Although conditions (2) and (4) are not satisfied, the tremendous effect of the almost complete lack of energy-dissipation capacity and the dynamic effects produced by condition (3) predominate. As a result, the stability of glass is very poor; for almost any size of specimen, instability is assured. To be able to detect some stability, the specimen must be reduced to a very small size (on the order of 1 mm, as is shown in Section IX), which will thus become the glass *transition size* (defined in Section XI).

It is known that other factors—e.g., temperature, strain rate, and triaxiality of stress—also affect the brittle-ductile transition. The triaxiality factor has a direct bearing upon the present discussion because, in most cases, fracture begins at a notch (natural or artificial, on the surface or internal) that creates triaxiality, and hence adds brittleness to the material. It must be recognized, therefore, that the true transition sizes are somewhat higher than the observed ones in all such cases.

Instead of considering the transition sizes of different types of materials, as was done here, materials may be classified according to their properties for *the same size* of a specimen (say, laboratory size) in the following manner: Mild steel is ductile because it is on the small side of the transition, and the instability condition is satisfied after the initiation condition has been satisfied.[2] In other words, a crack will be initiated, but it will remain stable. Concrete is semiductile because it is within the transition—in certain cases (depending upon the mode of loading), the initiation condition is satisfied first, whereas in other cases, the instability condition occurs first. Glass is brittle because it is on the large side of the transition; therefore, the instability condition is always the first to be satisfied. When a crack initiates in glass, it is immediately unstable.

Examples of the strain-energy size effect in specific materials within the three broad groups described above are presented in Sections VI–IX.

VI. Size Effects in Metals

That specimen size influences the strength and ductility of metals (the more brittle metals in particular) has long been recognized. Several explanations have been proposed to account for this phenomenon. These may be categorized into three main groups: (1) flaw statistics, (2) technological causes, and (3) stress multiaxiality.

Group (1), which includes the several variations of the flaw-distribution function, suffers mainly from an inability to account for the observable effect *after* crack initiation; i.e., during the propagation stage.

Group (2) includes all of the variations in the material properties that result from the manufacture of specimens of different sizes. The main deficiency of this explanation is that, in most cases, the same size effect is observed when the specimens are machined from the same stock.

Group (3) includes all cases in which size variations alter the load condition from plane stress to that of plane strain, thereby affecting the axiality of stress, which is known to affect the capacity to shear. The most familiar example is the thickness effect, the main weakness of which is that it exists only for changes from very thin to somewhat thicker plates, but not beyond this range. Also, the thickness effect is nonexistent in cylindrical specimens, where the situation is one of plane strain regardless of size.

Several examples of size effects in metals, in which the effect cannot be explained solely by any of the recognized theories, will now be considered.

As early as 1932, Docherty (Ref. 18) reported results of static bending tests on geometrically similar, notched, cantilever beams. Their sizes varied from 4 to 12 mm square (corresponding spans: 30 to 90 mm); three types of steel were used: 0.25 C as-rolled, 0.25 C normalized, and 3% Ni normalized. His criterion of ductility was the absorbed energy as determined from the area under the load–deflection curve to failure. In 1935 (Ref. 19), he also made similar tests on centrally loaded beams of mild steel (as-rolled and forged), with sizes varying from 10 to 100 mm square and from 2.5 × 10 mm to 25 × 100 mm. In all of these cases, there was a strong size effect, with absorbed unit energy decreasing with increasing specimen size. This decrease of energy extended beyond the specimen size of 100 mm cross section; thus it cannot be attributed to the change from plane-stress to plane-strain conditions (except for the zone of small widths, where the observed effect was indeed stronger).

The flaw-statistics factor could also be only a secondary cause in this case, because all beams were notched. In a notched beam, the fracture cross section is predetermined; therefore, one dimension (the length) is eliminated from the volume effect upon the flaw population. Another dimension (the depth) is eliminated by the very nature of a bending test, which requires that the fracture initiate at the tension surface. Thus, only the effect of the width remains. However, as is shown below, the results of other investigators indicate that varying the width alone causes a much lesser size effect than does varying all three dimensions. That flaw statistics could not be a dominating factor in this case is also evident from the shape of the load–deflection curves, which clearly indicate the development of stable cracking under

[2]The definitions of the various conditions related to the process of fracture are presented in Section X.

the notch. Flaw statistics can influence only the initiation of a crack, and not the events that follow.

In 1947, Davidenkov, Shevandin, and Wittman (Ref. 20) reported results of static bending and tensile tests on unnotched, high-phosphorous (0.52 P) steel. The cylindrical specimens varied from 1 to 16 mm in diameter, and breaking strengths were determined at a temperature of $-190°C$. Within the range of sizes tested, the decrease of bending strength with increasing size was 23%; the corresponding decrease in tensile strength was 26%.

Because these specimens were unnotched and the metal, in contrast to the metal in the preceding example, was very brittle (temperature was also very low), so that stable cracking could only have been limited, it may be assumed that both the flaw-statistics and the strain-energy factors played equal parts in the effect. The stress-multiaxiality effect is ruled out completely with such a brittle material and specimens of such a shape.

Also in 1947, Brown, Lubahn, and Ebert (Ref. 21) studied what they considered to be a section-size effect upon the static, notched-bar, tensile strength of Si killed, 0.25 C steel. In actual fact, they varied the size of geometrically similar cylindrical specimens so that length, as well as cross section, was varied. As noted above, the flaw-statistics factor in a notched specimen will affect only the cross section; however, other factors (that of strain energy in particular) are strongly influential because of the change in length. The diameters of the specimens ranged from 0.25 to 4 in. The notches were 60-deg, V-shaped, and covered 50% of the cross section. The results revealed a considerable decrease in notch strength–from 110,000 psi for the smallest specimen to 88,000 psi for the largest specimen. The corresponding decrease in notch ductility (defined as the contraction in area at the root of the notch) was from 20 to 2%.

These authors (see Ref. 21) presented a list of seven possible causes for the observed effect, the majority of which would qualify under category (2), above; namely, technological causes. In discussing these possibilities, they discarded them one after another, ultimately arriving at the conclusion that flaw statistics were the likely cause. The authors overlooked the possibility of the strain-energy effect. Indeed, this possibility becomes almost a certainty in view of their observation: "Examination of the fractured surface revealed two distinct regions. A central area of approximately circular outline had the appearance of a brittle (or cleavage type) fracture. Surrounding this area was a darker region in the form of a ring with its periphery at the root of the notch. This ring exhibited the characteristics of ductile (or shear) fracture. *The ratio of the brittle to the ductile area was found to increase with increasing section size.*"[3] In the terminology employed herein, this last statement would be expressed in the following manner: The transition from stable to unstable propagation is advanced with the increase of the strain energy content of the system.

In 1956, Schabtach and associates (Ref. 22) reported the failure of two generator rotors manufactured of nickel-molybdenum-vanadium steel. This steel showed some tendency to be brittle, but laboratory-size specimens indicated it to be sufficiently ductile for the intended purpose, and no significant loss of strength was expected, even in the presence of notches. Because the rotors constituted very large pieces of metal, however, it was considered necessary to study the effect of size upon this material.

Lubahn and Yukawa (Ref. 23) made the investigation in 1958 by performing notch-bend tests upon specimens of various sizes and notch radii. Their most important result is shown in Fig. 3. With increasing specimen size, the strength is always reduced. This effect increases with the sharpness of the notch, reaching the level of one-fifth of the strength of the small specimens for the worst combination of size and notch sharpness. Also observed was a tremendous decrease of notch ductility with increasing specimen size. The ductility apparently approached zero asymptotically for the larger specimens. These results were astounding, but the explanation offered by the authors, which has to do with flaw statistics, is totally inadequate for two reasons: (1) The beams were notched; therefore, the failure cross section was predetermined. (2) The notch ductility could not have had any connection with the statistics of flaw distribution, it being clearly the result of slow crack propagation. Other possible causes, such as hydrogen embrittlement, were considered by the authors, but were discarded as not likely to have been prime factors.

Among other observations made at that investigation (see Ref. 23), it was noted that all bars above a certain size broke suddenly, without warning, in the manner of crack propagation; in smaller specimens, short, discontinuous cracks could be observed prior to fracture. Because of these observations, coupled with the fact of the great effect of size upon both strength and ductility, it

[3]Italics added by the present author.

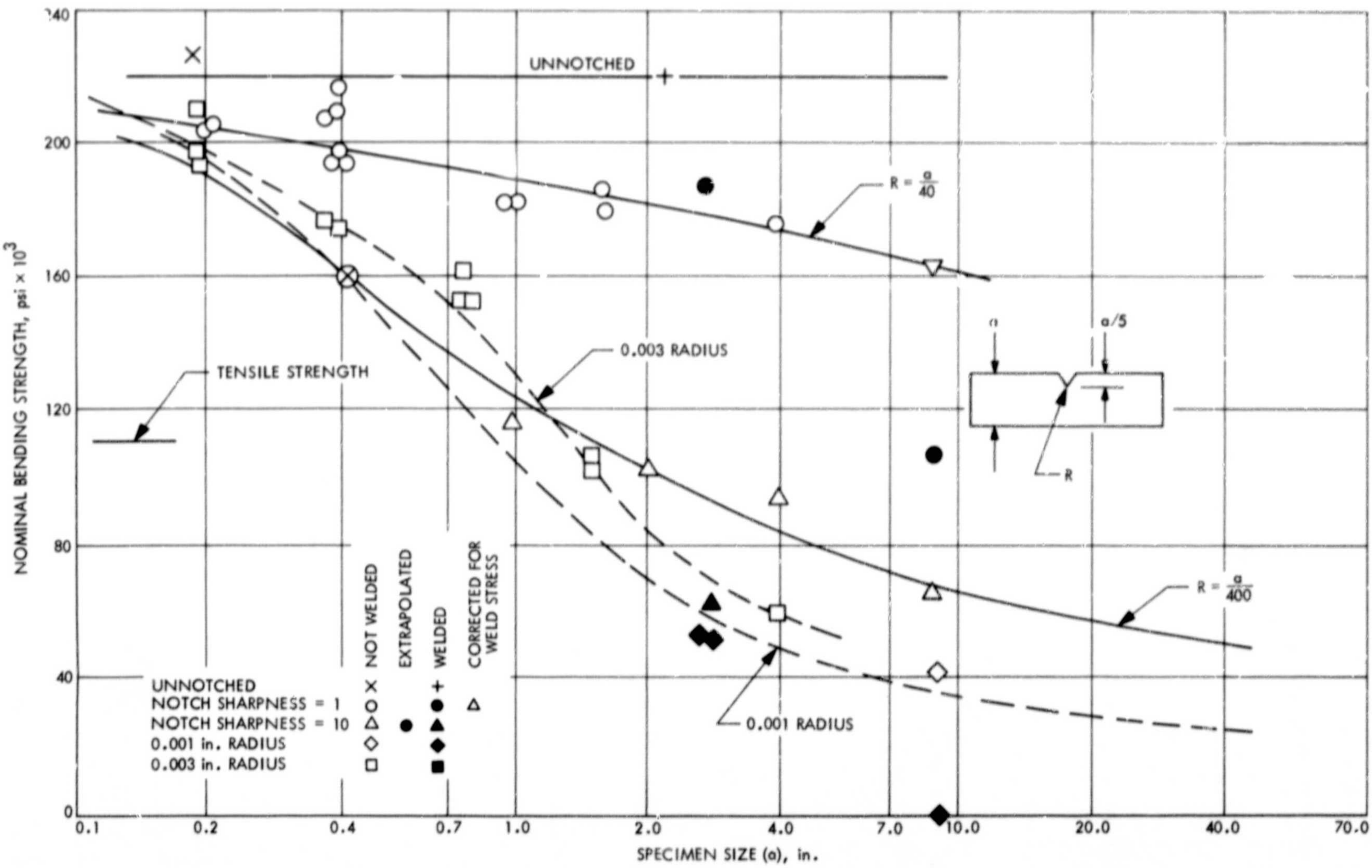

Fig. 3. Slow notch-bend strength vs size and notch sharpness for 20% notch depth (after Lubahn and Yukawa, Ref. 23)

is argued that the strain-energy mechanism could have been the main cause of this behavior. The contribution of the notch was to form a crack-type failure in the otherwise quite ductile metal by creating triaxiality of stress. Once cracking existed, the size of the specimen became a factor in determining the availability of strain energy, thus affecting the transition of cracking from stable to unstable. The combined effect of notch sharpness and specimen size, as shown in Fig. 3, is adequately explained by this new theory, as is the effect of size upon ductility when measured across the notch (actually recording the extent of cracking), and upon the other observations reported by Lubahn and Yukawa in Ref. 23.

The size effect was again observed by Lubahn (Ref. 24) in mild steel, a metal normally considered highly ductile as compared with the steel described above. When this metal was tested statically by means of notch-bend specimens, the notches induced sufficient brittleness at their roots for failure to take place by crack propagation. Changing the size of the geometrically similar specimens did not affect their ductility prior to crack initiation; however, it drastically affected the mode of crack propagation—particularly as regards the transition from stable to unstable cracking. This, in turn, was reflected in the ultimate strength, the cracking ductility (as opposed to precracking ductility), and generally in the load–deflection characteristics. Two typical curves are shown in Fig. 4, where the difference is evident both before and after the ultimate point. In the case of the small specimen, the stability depicted in the downgoing portion of the curve is the result of stress relaxation (as described in Section III).

Examination of the fracture surfaces corroborated the conclusions derived from the load–deflection measurements; namely, that for small specimens the extent of slow propagation (which appears fibrous in texture) had reached 50% of the section as against only 2% for the large specimens. The remaining cross section (i.e., where fast propagation occurred) showed typical cleavage texture.

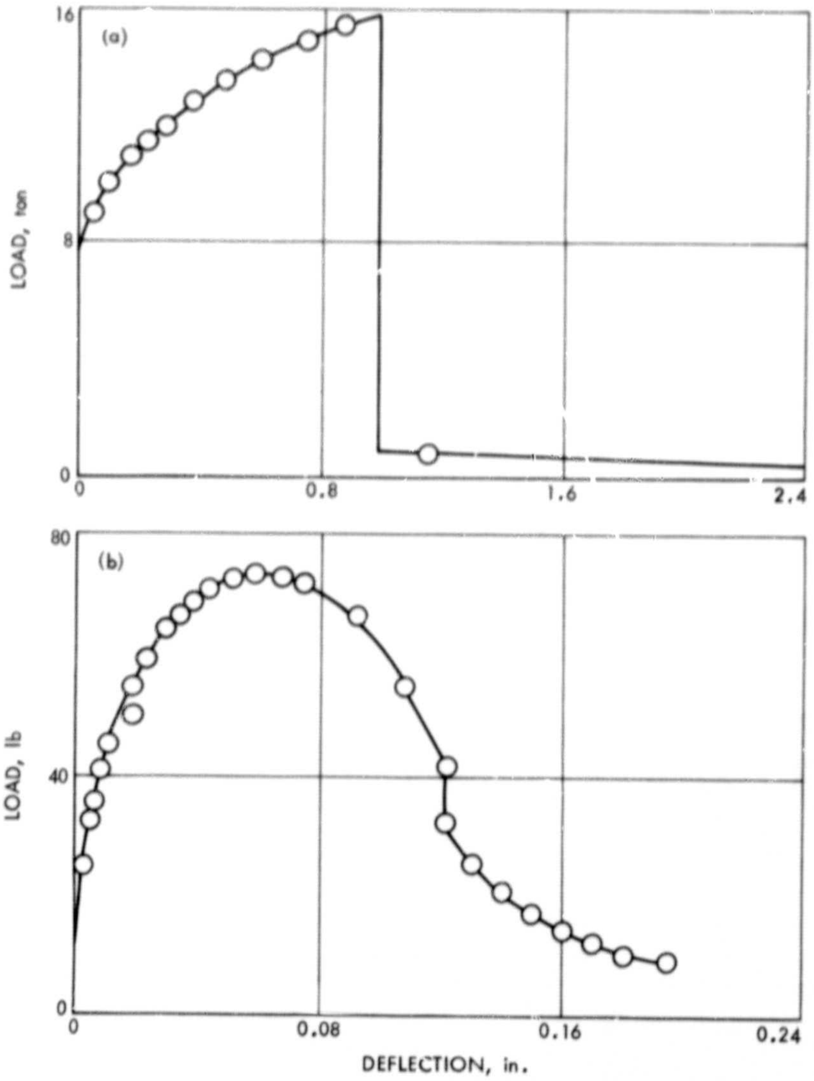

Fig. 4. Load–deflection curves for two mild-steel, V-notch Charpy specimens slow-bent at room temperature (after Lubahn, Ref. 24): (a) 2- × 2- × 11-in. specimen; (b) 0.008- × 0.008- × 0.375-in. specimen

Lubahn also tried to separate the effects of width, depth, and length by isolating them in turn. His results indicate that both width and depth must be large to cause the full size effect; apparently, a greater length is also necessary to obtain the full effect when a greater width is used. Increasing only the width apparently has only a small effect (see Refs. 18 and 19). It also appears that the effect of length is mainly upon the extent of spontaneous growth, according to the "load-relaxation stability" mechanism described in Section III. The effect of width is, to a great extent, through control of the degree of biaxiality of stress (i.e., the extent of "shear lips"). Without doubt, below a certain width, the appearance of shear lips is of major importance in preventing brittleness; however, the fact that size controls behavior well above this width points to the existence of another factor. Lubahn mentions stored energy in this context, but for some reason connects it only with the length of the specimens. Apparently he did not suspect that addition of energy in the other two dimensions might also have an effect.

A 4- × 4- × 22-in. specimen of mild steel still displayed some stable cracking, although mainly it underwent fast cleavage, whereas the nickel-molybdenum-vanadium steel described earlier had no stability above a section size of 0.4 × 0.4 × 2.2 in.

An extensive study of the effect of specimen size upon the static and fatigue strengths of various metals was conducted by Chechulin (Ref. 25). His study covered various types of steel, copper, and aluminum alloys. Generally, he found the following relationship: the higher the coarseness of the structural grain, the greater the size effect. A similar effect was produced by increasing inhomogeneities in the structure. Chechulin was mainly interested in fatigue, and the Russian practice is to notch specimens in fatigue studies; therefore, he introduced similar notches in his static-tensile specimens of various types of brittle steel alloys. The notch diameter was varied, and it was first discovered that these notches increased the strength. Then, by varying the specimens diameter from 6.5 to 12 mm, he found a reduction of the ultimate tensile strength of up to 13 kg/mm^2 for both notched and unnotched specimens. Corresponding to this loss of strength was an appreciable reduction in cross-sectional area and percentage elongation.

For notched copper and aluminum specimens, he found that increasing the diameter from 5 to 40 mm reduced the proportional limit by 60 and 13%, respectively. The ultimate strength was almost unaffected. For MA-5 aluminum alloy, the difference in proportional limit between specimens of 10- and 40-mm diam was 45%. In V-95 and D-16 aluminum alloys, the effect was smaller; namely, 10 to 20%.

The fact that the same quantitative size effect was observed for both notched and unnotched specimens tends to eliminate the flaw-statistics factor as the main cause. In the notched specimens, because the cross section of failure is fixed (probability of failure being unity), only the area is changed with diameter; in the unnotched specimens, the entire volume is changed. The observations connecting the size effect with the coarseness and heterogeneity of the structure support the strain-energy theory suggested in this report. Both of these properties tend to promote larger dynamic effects, and thus would enhance sensitivity to size.

Very convincing evidence of the existence of the strain-energy size effect in metals (three different types of mild steel) was reported by Fearnehough (Ref. 26) in 1963. He employed the drop-weight test on V-notched beams to determine their brittle–ductile transition temperature. To make this temperature independent of specimen size, he used a normalization procedure whereby the beams were allowed to bend to varying angles so that the notch strain was the same for each specimen size. Despite this precaution, the results indicated a strong dependence of transition temperature upon specimen size, large specimens having a higher transition temperature than small specimens.

This, of course, indicated that increasing size was accompanied by an increase in brittleness. The effect was demonstrated for each dimension—length, width, and depth—of the beams. Further to support his idea that this was a strain-energy effect, Fearnehough superposed a longitudinal tensile stress on one of the small beams (7 × 0.5 × 0.5 in.) while executing the drop-weight test. The transition temperature for this sample was increased; in fact, it was found to be equal to that of a larger sample (14 × 2 × 1 in.). Without this superposed tension, the difference between the transition temperatures of these two specimen sizes was about 30°C. This observation clearly supports the theory presented herein because the addition of a tensile stress, although it increases the strain-energy content, does not change the distribution of flaws or alter the metallurgical constitution of the material.

In summary, the following can be said for the size effect in metals:

(1) It can be observed in a great number of metals from the viewpoint of ductility and brittleness.

(2) In the more brittle metals, it is immediately observable; in the more ductile metals, a condition must first be fulfilled to convert the otherwise ductile failure to one of crack propagation, e.g., a notch (i.e., triaxiality of stress), high strain rate, or low temperature.

(3) It is more pronounced in metals with coarse grain structure or a high degree of heterogeneity.

(4) It is manifested in breaking strength, in ductility accompanying fracture (i.e., "semiductility," as defined in Section V; in Ref. 10, it is called "second type ductility"), in the area under the load–deformation curve, in the brittle–ductile transition temperature, and (in certain cases) by the proportional limit.

(5) The three dimensions of the specimen contribute to the effect, but the relative importance of these contributions has not yet been determined.

(6) Although flaw statistics may be a factor in the size effect (particularly in the very brittle metals), its influence is limited to initiation only. Therefore, in all cases where there is evidence of slow crack growth or of accompanying ductility, statistical effects are negligible. The fact that the size effect in notched specimens of brittle metals is almost the same as it is in unnotched specimens suggests that, whatever effect flaw statistics has upon initiation, it is small in magnitude.

(7) Multiaxiality of stress is a factor in thickness effect, but only within the limited range where the conditions change from plane stress to plane strain.

(8) Evidence in favor of the strain-energy factor is as follows:

  (a) The fracture surface clearly shows the transition from stability to instability; moreover, it shows the dependence upon size of the location of this transition.

  (b) Visual evidence of stable cracks exists in small specimens, but is absent in large specimens.

  (c) Slow cracking in small specimens is evidenced by the increased curvature of the stress–strain curves before these curves reach their summits.

  (d) Fracture in large specimens is explosive and unannounced; in small specimens, fracture is accompanied by a low, tearing sound.

  (e) The size effect is dependent upon the coarseness and the heterogeneity of the structure.

  (f) Superposed tensile stress affects the transition temperature of V-notched bend specimens, as described above.

(9) The level at which size ceases to affect strength (or "semiductility") varies with different materials. Broadly, this level increases with the ductility of the material, but interfering factors—e.g., triaxiality (notches), temperature, and strain rate—may obscure this tendency.

(10) The worst combination of notch sharpness and size reduces the strength of a certain steel from 220,000 to 40,000 psi (see Fig. 3). According to the theory

proposed herein, the strength of an unnotched beam will also be reduced to the above level if the beam is sufficiently large. This should scare bridge designers. In reality, the danger is not so acute because of two factors: (a) Real structures are always highly redundant; therefore, as soon as a fracture initiates, the material surrounding the crack is unloaded because the load is transferred to alternate members. (b) In real structures, large, monolithic elements are very seldom used. The various types of joints make the structure safer by acting as drains for elastic energy.

VII. Fatigue–Size Effects in Metals

It has been known for nearly half a century that the fatigue strength of metals increases appreciably with decrease of test-specimen size, but no satisfactory explanation has yet been offered. As reported by Grover (Ref. 27), a few examples (Refs. 28–30) are listed in Table 1. It is clear that the effect of size upon these unnotched specimens is very pronounced, and that it increases with increase of impurities content. Moore (Ref. 31) reported an effect upon two types of steel—SAE 4340 and SAE 1035 (Fig. 5). The first steel (SAE 4340) showed a rotating–bending fatigue limit decreasing from 75 klb/in.^{2} for a ⅛-in.-diam specimen to 65 klb/in.^{2} for a 1-in.-diam specimen, and not decreasing for diameters above this size. The second steel (SAE 1035) decreased from 40 klb/in.^{2} for a ⅛-in.-diam specimen to 35 klb/in.^{2} for a 1-in.-diam specimen without further change. On the basis of the above and other results that he cited, Grover reached the following conclusions:

(1) There is little size effect in axial fatigue loading.

(2) There is a considerable size effect (varying with the material) in bending and torsion.

(3) Large notches in large specimens cause more reduction in fatigue strength than do geometrically similar notches in smaller specimens.

In addition to the above-mentioned data, Moore and Morkovin (Ref. 32) reported fatigue–size effects in carbon-steel cantilevers as follows: fatigue strength is reduced by 15 to 20% with increase of specimen diameter from 3 to 30 mm. Draigor and Val'chuk (Ref. 33) presented many data (mainly obtained in the U.S.S.R.) concerning size effects in fatigue. They attributed the phenomenon to three possible mechanisms—flaw statistics, strain energy, and technological factors.

Table 1. Rotating–bending fatigue strengths of large-diameter steel shafts

Material	Diameter, in.	Fatigue strength,[a] klb/in.^{2}	Reference
0.4 to 0.5% C	0.3	32.4	28
	1.5	28.0	
	7.0	14.5[b]	
NiCrMo	0.5	56.0	29
	9.0	38.2	
0.22% C, acid open hearth	0.4	31.0	30
	4.9	28.5	
0.22% C, electric furnace	0.4	39.2	30
	4.9	33.4	

[a]At 10^7 to 10^8 cycles.
[b]Considerable scatter.

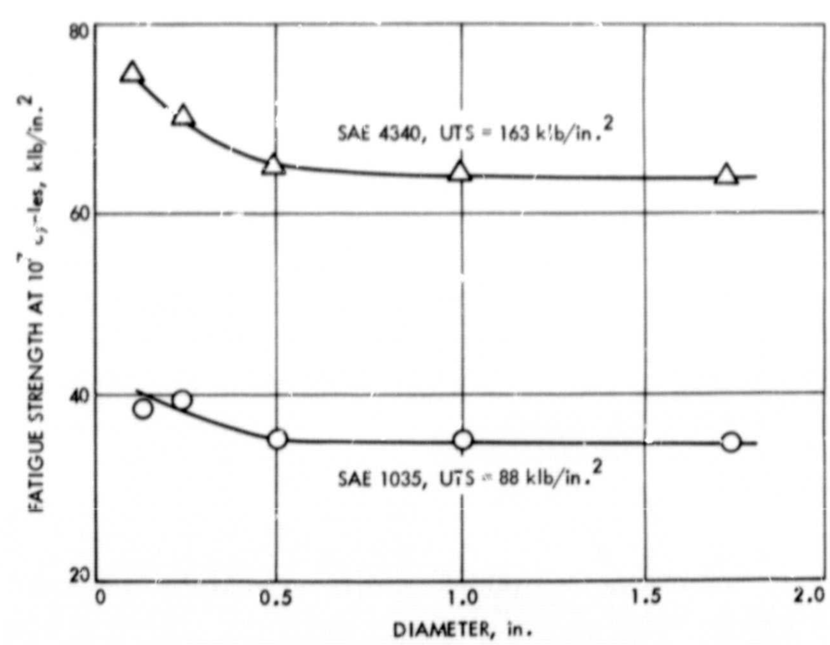

Fig. 5. Size effect in unnotched steel specimens in rotating–bending fatigue (after Moore, Ref. 31)

Fatigue can serve as an excellent example of the hypothesis offered in this report to explain size effects because slow crack propagation is the main characteristic of fatigue behavior. Some mechanism, as yet not fully understood, causes a crack to propagate slowly with the number of load cycles until it reaches a critical size, at which point the crack suddenly runs and causes complete fracture. The present theory contends that, to a great extent, the strain-energy content of the system controls the onset of this critical event in a way very similar to that described earlier for static loading. This size effect is always present in fatigue loading because slow cracking is always part

of fatigue, whereas this is not so in static loading. The following observations support this view:

(1) A fatigue fracture surface always reveals the transition from slow to catastrophic growth. The proportion of the slow-growth area increases with decrease in specimen size.

(2) The effect increases with increase of impurities content.

(3) The effect is much more pronounced in bending and torsion than in axial fatigue.

The third observation deserves consideration. Bending and torsion modes differ from the axial mode in the manner of stress distribution, there being a gradient in the bending and torsion modes and uniformity in the axial mode. Cracks caused by bending and torsion initiate on the outer fibers of the material, where stress is maximal; consequently, these require a higher number of load cycles to penetrate the bulk of the material and reach instability than do their uniaxial counterparts. Therefore, changing the specimen volume—and thus advancing or retarding the transition from stable to unstable cracking—has less effect in an axially loaded specimen, where the number of cycles from nucleation to instability is smaller.

In any case, the flaw-statistics theory cannot account for this effect for the following two reasons:

(1) The logic of this theory would require the size effect to be less pronounced in bending and torsion than in an axial test because the location of cracking is to a great degree predetermined (especially in bending), whereas no such bias exists in the axial specimen.

(2) From the appearance of the fracture surface, it is obvious that the effect of size is manifested mainly during the slow-propagation stage, and not in that of initiation.

In fatigue tests, as in static loading, the size effect is limited in the sense that every material has a limited range of sizes that affect its fatigue life. However, it should be noted that, in fatigue tests, this transition size is much smaller than it would be in static loading of the same material. For example, Fig. 5 shows this size to be between approximately ⅛ to ½ in. for *unnotched* specimens. In static loading of similar material, the transition will be observed at this size range only if *notches* are introduced (i.e., if the brittleness is increased). This fact is consistent with theories that attribute embrittling effect to cyclic loading. In other words, cyclic loading nucleates a crack that is the equivalent of a notch in a static test.

VIII. Size Effects in Concrete-Type Materials

Concrete is a material used almost exclusively to sustain compressive loads. Consequently, most of the information concerning its strength refers to compressive strength. It has long been a matter of common knowledge that the size of the test cylinder[1] exerts a marked influence upon the indicated strength. An early investigation into this problem was made in 1925 by Gonnerman (Ref. 34), who varied the diameters of the cylinders from 1.5 to 10 in. For such an increase of size, he obtained a reduction of compressive strength of up to 20%.

Another early investigation was conducted by Blanks and McNamara (Ref. 35) in 1935. They varied the size of the cylinders from 2 × 4 to 36 × 72 in., ensuring a minimum diameter to maximum aggregate size ratio of four; they also measured both elastic constants E and ν, as well as the ultimate compressive strength. No effect of size upon elastic constants was observed, but there was a marked effect upon strength, as shown in Fig. 6 (strength of 6- × 12-in. cylinders was taken as 100%). The effect of size upon strength was found to be independent of the size of aggregates and age; therefore, the results of various mixes were averaged and included in the curve. Their report includes neither stress–strain characteristics nor statistical scatter of results, and the authors found no satisfactory explanation of the size effect.

No doubt the flaw-statistics factor played a role in the initiation of fracture. However, in a compressive test of a material such as concrete, cracking commences when the load is as low as 30% of the ultimate load; during the upper 70% of the loading, therefore, the process is one of crack propagation. The weakest-link theory has very little bearing in such a case. By contrast, the strain-energy effect must be strong during the very pronounced stage of crack propagation.

In 1941, Tucker (Ref. 36) applied statistical theories to beam-test results reported earlier by several experimenters with concrete: Abrams (Ref. 37), Reagel and Willis (Ref. 38), Gonnerman and Shuman (Ref. 39), and Kellerman (Ref. 40). Tucker divided his study into the

[1]U. S. standard test specimens are cylindrical, with a diameter-to-height ratio of 1:2. European specimens are cubic.

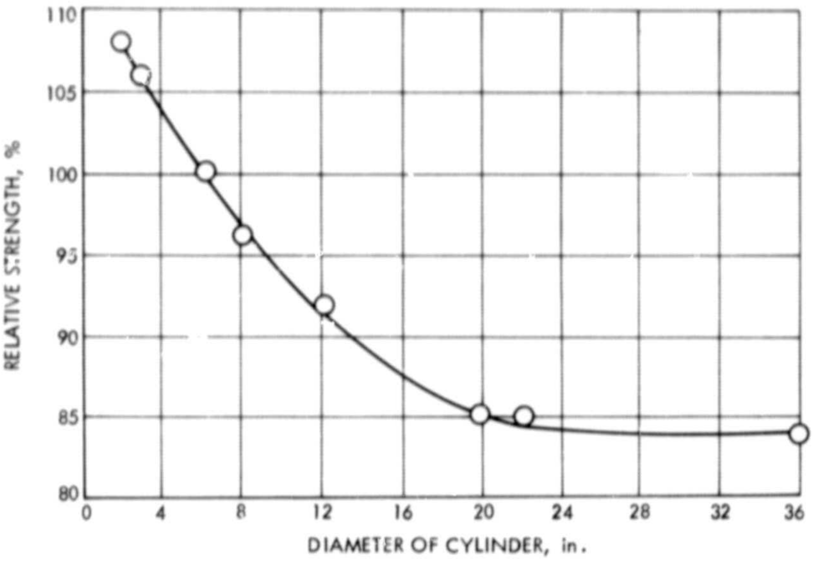

Fig. 6. Effect of size of test cylinder upon relative compressive strength of concrete—diameter:height ratio = 1:2, reference cylinder = 6 × 12 in. (after Blanks and McNamara, Ref. 35)

separate effects of length, depth, and width. For the length of a constant-bending-moment beam, he cites Reagel and Willis (see Ref. 38), who found a 1.4% reduction in strength upon doubling the specimen length. Tucker's prediction, based upon the weakest-link theory, was 2.7%. The data of Gonnerman and Shuman (see Ref. 39) concerned cantilever, center, and third-point loaded beams varying in length up to five times the shortest length. The modulus of rupture of the beams loaded at third points was unaffected by length changes, but that of the beams centrally loaded and loaded as cantilevers showed a definite reduction with length. Although Tucker does not attempt to explain this point, the experiments seem to contradict the theory. Where the zone of fracture is more or less predetermined (as in a center-point load or a cantilever), the size effect was definite; where the site of fracture was subject to chance (as in third-point loading or a constant-bending-moment beam), the size effect was smaller than predicted or even nonexistent.

These observations are, however, in accord with the proposed theory. When the strain energy is concentrated in a small volume of the material, the specimen can be regarded as "small" in the sense of the present hypothesis. Strain-energy concentration occurs in cases of center or cantilever loading because of the gradient of the bending moment. The concentration is even greater in the case of a notch, but only for materials such as concrete or glass; in mild steel, a notch will introduce brittleness, which has an opposite effect (equivalent to an increase in volume). A "small" specimen thus favors the stable propagation of a crack, and enables the development of a size effect during this propagation. This hypothesis is borne out by the familiar observation of a sudden and explosive fracture in a third-point-loaded beam vs a more controlled, slow fracture under a center-point load.

For depth effect, Tucker (see Ref. 36) also used the weakest-link theory, with some modifications, to allow fracture to initiate at an inner fiber (otherwise, based upon probabilities, there would have been no size effect). Despite this modification, his prediction was much lower than the experiments showed. Reagel (see Ref. 38) demonstrated a reduction in strength of 11.5% for an increase in depth from 4 to 10 in.; Tucker's prediction was 4%. It seems reasonable to assume that at least the 7.5% difference is due to the strain-energy effect. Abrams (see Ref. 37) presented data showing a 6.5% reduction in the modulus of rupture, with an increase in depth from 4 to 10 in. for third-point-loaded concrete beams.

For width effect, Tucker (see Ref. 36) suggested a statistical theory that he termed the "summation" theory. According to this theory, the strength of a specimen is equal to the sum of the strengths contributed by the component elements (i.e., a parallel arrangement of elements). This was necessary to account for the lack of a width effect (regarding strength, but not regarding dispersion) in the data presented by Reagel and Willis (see Ref. 38). Their results showed that beams with widths of 4, 6, 8, and 10 in. failed at mean strengths of 801, 813, 816, and 817 psi, respectively. The standard deviation values were 6.3, 4.5, 4.5, and 3.8%, respectively. Gonnerman and Shuman (see Ref. 39) also tested width effect, but their results were very erratic. Again, there was no indication that width affected strength. Once more, therefore, as in the case of length effect, one is faced with the phenomenon of no sensitivity to size—a behavior totally unacceptable to supporters of any of the statistical theories. No modifications of these theories, such as Tucker's "summation" theory, will reconcile this basic contradiction.

On the other hand, judging this phenomenon from the standpoint of the strain-energy theory, it does not present a contradiction because this theory does not require size effect to manifest itself in all cases. It was noted previously that mild steel does not show size effect unless a notch (or perhaps low temperature or high strain rate) is introduced to impart some brittleness so that the failure will become one of crack propagation. Only with fulfillment of this prerequisite (which can also be regarded as the shifting of the material to "larger" volume or toward

its transition size) will mild steel exhibit a size effect (see Ref. 23). In the present case, concrete will similarly show no size effect if it is far from its transition size. In the two cases where no width effect was observed, the specimens were too "large"; their size should have been reduced to permit slow crack development so that size could have had an effect. One way of doing this is to introduce a notch,[5] which will concentrate most of the strain energy in a small volume near its root. Cohen (Ref. 41) did exactly that, and obtained a very pronounced width effect, using center-loaded beams with notches at the center of their tension faces. Decreasing the width of the specimen from 7 to 1 in. increased the rupture modulus by 50 to 100%, depending upon the span. However, the number of beams tested was not sufficient to determine the dispersion of results.

In 1952, Wright (Ref. 42) conducted a series of tests with concrete beams and varied all three dimensions (without, however, isolating the effect of each). Increasing the beam size from 3 × 3 × 9 in. to 8 × 8 × 24 in. caused a 28% strength reduction for a case of third-point loading and 33% for a case of center loading. However, these values also included differences caused by the different loading rates (about 9%) and what Wright believed to be a difference in quality between the material of beams 3 in.^{2} in section and that of beams 8 in.^{2} in section (11%). The latter idea is totally unacceptable; the error probably stemmed from his resorting to sawing large beams into small ones for the purpose of proving the difference in quality—an act that must have affected the surface of the tension face. It is much more likely that the 11% difference (perhaps even more) was caused by the strain-energy effect. (The balance of about 8% was, according to Wright, a statistical effect.) The scatter of results was qualitatively as anticipated on statistical grounds; i.e., decreased for larger specimens, but to a smaller degree. It should be noted that, according to the strain-energy theory, scatter should *increase* somewhat with the increase of specimen size because the increase of strain energy advances the fracture towards its initiation, and initiation is subject to a less favorable distribution function (namely, one of extreme values). Wright's observation (see Ref. 42) may, therefore, be an indication that the two opposing effects—flaw statistics decreasing scatter and strain energy increasing it—were operative concurrently.

[5]This may seem to be a contradiction. In mild steel, a notch brings the material nearer to instability, whereas in concrete (and more so in glass) it brings the material nearer to stability. However, this is no contradiction, as will be shown in Section IX.

Considerable support of the strain-energy theory and detraction from the flaw-statistics theories were provided by Glucklich (Ref. 43) in 1957. He showed that the inclusion of a steel spring in series with the specimen has an effect similar to an increase in the specimen size—i.e., a pronounced effect upon strength, shape of the load–deflection curve, and dispersion of results. The addition of the spring simulates an increase in specimen size in that it increases the strain-energy content of the specimen, but does not enlarge it from the viewpoint of statistics (i.e., the flaw population is unaltered).

Hardened, cement-paste cylinders, when compressed in an ordinary hydraulic press in series with a steel spring, failed at levels as low as 35% of the failure load without a spring. The load–deformation curves were almost straight lines up to failure, as compared with marked nonlinearities above a certain level when no spring was used. The mode of fracture was a single, clear-cut separation, as compared with total disintegration in the case of no spring, and the coefficient of variation of strength values was increased threefold. Researchers in concrete know well that, at about 35% of the compressive strength, internal cracking commences; Glucklich's observation, therefore, suggests that the added strain energy simply advanced fracture to coincide with the onset of internal cracking. This made an early crack (perhaps the first) become unstable, whereas otherwise it would have become stabilized. This unstable crack then led to total fracture. The truncated load–deformation curve, the single-crack mode of fracture, and the increased scatter all point to the validity of this assumption. It is not contended, of course, that the added spring is quantitatively equivalent to any particular known size increase of the specimen. This problem is complex. As should be clear from Section V, the addition of a spring not only shifts the dynamic-response curve of the system, but also alters this curve completely. In fact, with a concrete-type specimen, it is quite likely that the response of the spring completely dominates that of the system. A strong case has, however, been made for the strain-energy theory.

In 1962, J. P. Romualdi[6] conducted tests with variable-span concrete beams reinforced with closely spaced, short steel wires. For halved spans, he obtained fracture stresses exceeding those of full beams by 35 to 100%.

In 1966 and 1967, Glucklich and Cohen (Ref. 44) extended Glucklich's earlier work to cover other states of

[6]In an unpublished report (1963).

stress and used a different material. The following tests were conducted using plaster of paris:

(1) Unnotched plates, with and without a spring in series with the specimen, were loaded in tension. The spring caused a mean reduction of 29% in the fracture stress and a 50% increase in its coefficient of variation.

(2) Compression cylinders, made of a different type of plaster of paris, were loaded—again, with and without a spring. The corresponding figures attributable to the spring were 30 and 42%, respectively.

(3) A large number of beams, in which shallow notches in the center of the tension face were made during casting, were center-loaded and tested for length effect. The notch was used to predetermine the cross section of failure, thus completely eliminating the statistical effect. Doubling of the span caused a mean strength reduction of 19%, but dispersion values were inconsistent, sometimes increasing and sometimes decreasing for the double-span beams.

(4) Notched beams, as described above, were tested for length effect over a much wider range of spans. Quadrupling of the span resulted in reduction of the fracture stress by approximately 30%, as shown in Fig. 7. Again, no correlation was observed between length of specimen and dispersion of strength results.

(5) Other similarly notched beams were tested for the effect of a spring mounted in series with the center-point load. In addition, the deformation across the notch was measured on the tension face. The spring reduced the breaking load by 11.5%, and the load-deformation behavior deviated very little from the linear. Without a spring, the deviation was much more pronounced. The coefficient of variation of the breaking loads was 1.35% with a spring and 1.12% without.

The above observations seem to show conclusively that a strain-energy size effect exists over and above the statistical effect. This is indicated by tests (3) and (4), above, where the statistical effect was eliminated by the introduction of notches. In fact, the very strong observed effect suggests that the statistical effect, when it exists, is very small in relation to the strain-energy effect.

The above observations also support the hypothesis concerning the advancement of instability caused by strain energy. This is manifested by the disappearance of most of the nonlinearity in the load–deflection curve,

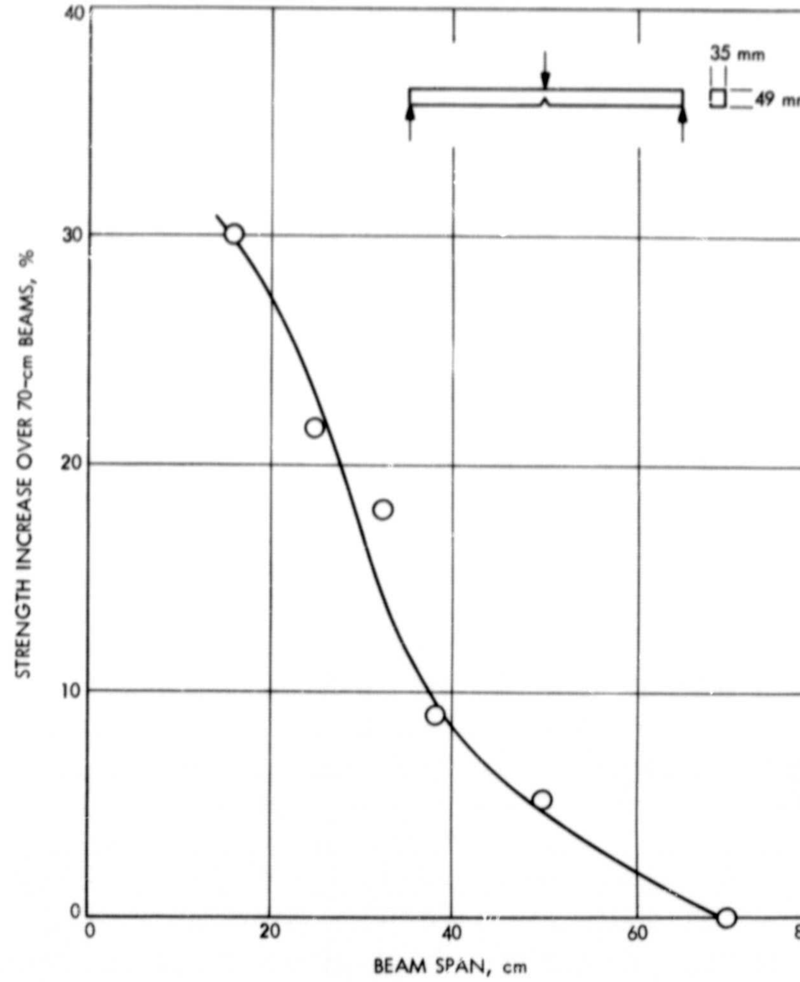

Fig. 7. Span dependence of the rupture modulus of a notched plaster of paris beam (after Glucklich and Cohen, Ref. 44)

and by the increase of dispersion with increasing strain energy or size. In tests (3) and (4), the apparent lack of effect upon dispersion in the two cases of span variation, as compared with the positive effect in the case of a spring, is not understood. It may, however, indicate that some statistical effect occurred even in the notched beams because (as has been explained) the statistical dispersions and strain-energy dispersions are opposite. Where a spring was used, of course, only the strain-energy effect was operative.

IX. Size Effects in Glass and Glassy Polymers

Glass is the classic material in which size effect was predicted on the basis of statistical considerations. Griffith (see Ref. 1) was the first to use this method of prediction, and several investigators—Fisher and Holloman (Ref. 45), Gibbs and Cutler (Ref. 46), and others—offered various flaw-distribution functions to account for the different observations. All of the theories predict that larger specimens will be weaker; quantitatively, however, there

were and still are unexplained discrepancies. In the discussion that follows, observations will be cited and reviewed in the light of the theories offered by the investigators, along with the theory presented in this report.

As long ago as 1891, Auerbach (Ref. 47) discovered the law, now known as Auerbach's law, which states that the ring cracks produced in glass under the pressure of spherical indenters are the result of stresses that depend upon the size of the indenters. A size effect exists in the sense that the smaller the indenter, the higher the critical stress, as computed on the basis of the Hertz equations (see Timoshenko, Ref. 48).

In the years that followed, several investigators—Andrews (Ref. 49), Longchambon (Ref. 50), Tolansky and Howes (Ref. 51), and Tillet (Ref. 52)—repeated and extended Auerbach's work. All of them observed the following: Under the pressure of a steel indenter (applied statically or by impact), glass breaks by a ring crack encircling the area of contact, usually at a short distance from its boundary. The fractures caused by indenters of varying radius were geometrically similar and, in all cases, Auerbach's size effect was observed. The various empirical relationships between the indenter radius and quantities that measure the critical stress were reviewed by Roesler (Ref. 53), who also proved their equivalence.

Thus, in works of different investigators, the validity of Auerbach's law has been established for about 15 different types of silicate glass. Mathematically, this law states that P/r = constant, where P is the force between the indenter and the glass and r is the indenter radius; a constant breaking-stress criterion would require $P/r^2 = A\sigma^3$ = constant (A being a constant) in accordance with the Hertz equations. The range of indenter radii for which this law was confirmed was 0.5 to 15 mm. Tillet (see Ref. 52) was the first to extend the range to 125 mm. She, too, confirmed that P/r = constant for spheres from 1.5 to 35 mm, but she found the law P/r^2 = constant for spheres above 35 mm; i.e., the ending of the size effect. Tillet's results are shown in Fig. 8. She also noted the following additional observations:

(1) The critical tensile stress varied by a factor of two with the variation of indenter size from 1.5 to 35 mm.

(2) The suddenness of the appearance of the cracks increased with the radius of the indenter. For small indenters, the crack could be seen to propagate in a circle, with a speed on the order of 1 mm/s. Furthermore, when pressure was applied more slowly, the crack velocity became slower.

(3) The scatter of force observations was the same for both small and large indenters.

(4) When the glass surface was scratched with a diamond, for indenters below 1 in. radius (i.e., in the range of size effect), no difference was found in breaking force between scratched and unscratched glass; for larger indenters (i.e., in the range of no size effect), the breaking force was much less (30% with an indenter of 4 in. radius) in the scratched glass.

(5) For small indenters, the size effect in glycerol sextol phthalate (organic glass) was similar to that in silicate glass.

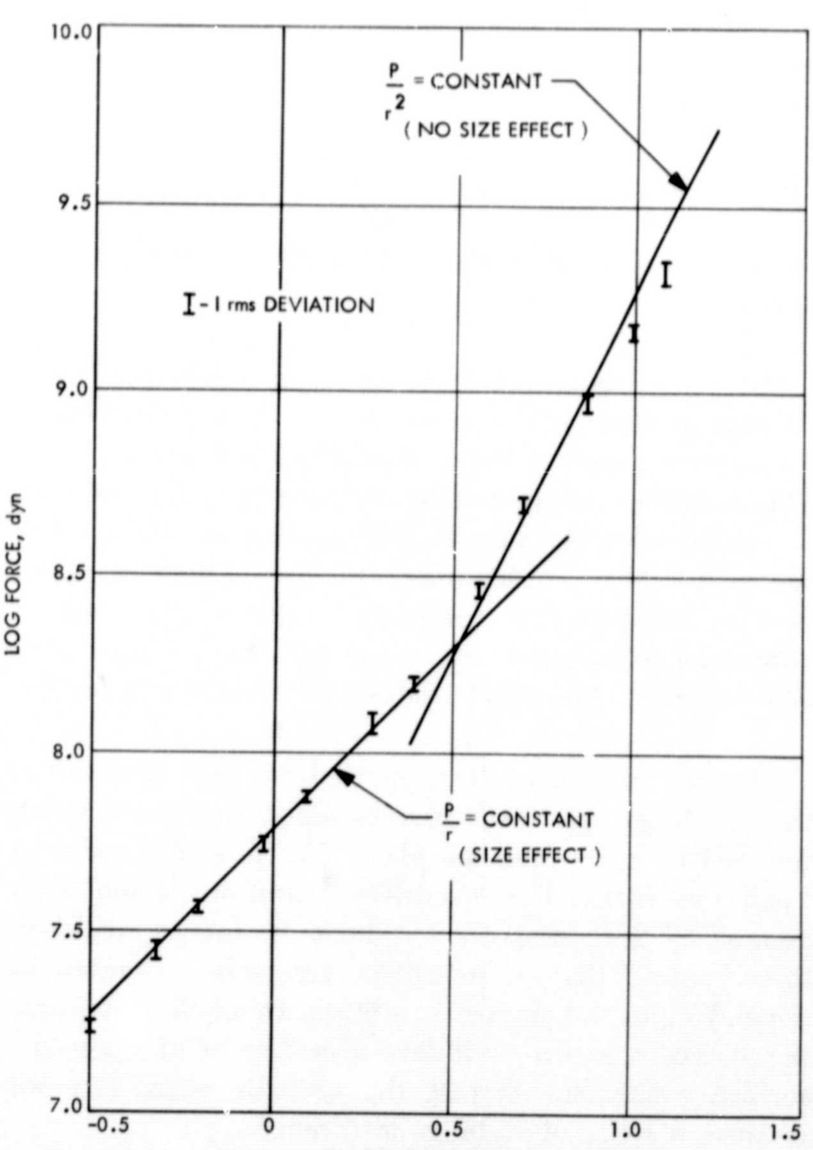

Fig. 8. Effect of indenter size upon force required to produce a ring crack in silicate glass (after Tillet, Ref. 52)

The flaw-statistics theory, which has traditionally been applied to the strength of glass, is invalidated here for the following reasons: (1) Scatter was not higher for the smaller indenters, as it should have been according to all statistical theories. (2) The degree of smoothness of the glass surface (whether scratched or not) made no difference in the effect of size on strength. Where it did make a difference, there was no size effect. (3) The effect was the same for silicate glass and for organic glass, although the states of the flaws were completely different. (4) The sudden ending of the size effect above a certain size (see Fig. 8) is inconsistent with any of the statistical theories. In addition, Roesler (see Ref. 53) advanced the following objections: (1) The flaw-statistics theory does not readily lead to the special power laws, which are true empirically. These laws would have to be attributed to a peculiar special flaw distribution; it seems improbable that this distribution should have accidentally existed in all of the samples of different types of glass that have been tested. (2) Reviewing the works of five investigators, Roesler did not find the scatter effect in any of their reports.

By contrast, the observations of Tillet and the other investigators are highly consistent with the theory proposed herein, as demonstrated in the paragraphs that follow.

For small indenters (equivalent to small specimens because of their geometrical similarity), the slow crack velocities indicate a stage of stable crack propagation. This is further supported by the fact that this velocity depended upon the rate of load application. With large specimens, the almost instantaneous growth suggests that the strain-energy instability condition was satisfied before the initiation of the crack; therefore, when initiation occurred, the crack immediately became unstable.

The last statement is supported by the observation that, in large specimens, the breaking force was much reduced by scratching the glass. This does not occur in small specimens. The instability condition having been assured by the size of the specimen, its failure condition then became that of initiation, and this is known to depend upon the surface condition. In small specimens, the initiation stress doubtless also depended upon the surface condition, whereas the ultimate stress did not because a stage of stability intervened.

The almost abrupt change from size dependence to independence is consistent with the assumption of a transition size above which the strain-energy instability is presatisfied. In such large specimens, therefore, the failure condition becomes that of initiation, as indicated above; actually, some size effect due to flaw distribution should be anticipated. Indeed, careful examination of Fig. 8 will show that the slope of the upper curve is not 2, as drawn by Tillet, but more nearly 1.8. Some statistical effect exists, therefore, but it is small in magnitude.

The flaw-statistics theories require higher scatter for smaller samples. The strain-energy theory requires just the opposite, as explained in Section VIII. The fact that no difference was observed in the scatter suggests that both flaw statistics (for initiation) and strain energy (during propagation) were operative concurrently.

A great number of investigations of size effects in glass fibers have been carried out since Griffith first conducted his classical experiments (see Ref. 1). Griffith showed that, for a few seconds, freshly drawn fibers are of high strength independent of their size—up to 0.02 in. diam and ranging from 220,000 to 900,000 psi for different types of glass. The size effect existed for aged glass, and Griffith suggested an empirical function for the size dependency that yielded a maximum strength of 1.6×10^6 psi by extrapolation to zero size. His work, therefore, clearly indicated size effect to be the result of flaws that develop in the glass during its aging. Consequently, most of the investigations following those of Griffith were aimed at determination of the flaw-distribution functions to fit the observations. A fit was sought in both the median-strength and the scatter-of-strength results. To this day, unfortunately, no agreement has been reached on either of these points because of the totally contradictory results reached by different investigators.

As far as length is concerned—e.g., in the work of Anderegg (Ref. 54)—good agreement was reached between length effect of 13μ fibers (over a factor of 300) and a statistical function suggested by Fisher and Holloman (see Ref. 45). Reinkober (Ref. 55) also observed similar behavior.

For a diameter effect, however, the position is not clear. Mould (Ref. 56) reported data that were grouped around two distinct strength levels, with high scatter of the low-strength distribution and low scatter of the high-strength distribution. This was in obvious contradiction to any statistical theory, but is in agreement with the strain-energy size effect proposed herein. Anderson

(Ref. 57), in an excellent discussion of this problem, in this connection mentions Mould, and also the work of Thomas (Ref. 58). For very small fibers (2×10^{-4} in.), Thomas found a coefficient of variation of only 1% for a median strength of 550,000 psi. Such low scatter for such high strength is inconsistent with any statistical theory. On these and similar observations, Anderson comments that the only way to reconcile them with the flaw-statistics theory would be to assume either a complete absence of flaws at 2×10^{-4} in. diam or a flaw population of uniform size—both unlikely assumptions.

Observations also do not agree with theory insofar as the strength of a material is concerned. Extrapolations of the various formulas to zero size result in strengths far in excess of any theoretically conceivable strengths.

Because of the contradictory observations, several theories of the size effect in glass fibers have been proposed. Anderson (see Ref. 57) reviews these theories, and discredits most of them, including the following: oriented structure, oriented flaws, "frozen-in" strains, high surface strength, and flaw statistics. His rejection of the latter is mainly because of the low scatter of strength data observed in most of the investigations. Anderson's inclination is to support a flaw-statistics theory modified to account for different quenching times for fibers of different diameters (the latter factor affecting the flaw density of the fiber). He concludes, however, by stating that he regards the problem as unsolved.

In a similar review of the problem, Hillig (Ref. 59) considers the most serious objection to the flaw-statistics theories to be the various studies that show no diameter effect at all. He cites a number of investigators who had removed all flaws from the surface of the fibers by etching, lacquering, or liquefying, and consequently had obtained strengths independent of diameters up to ¼ in. In a similar manner, Hillig had obtained a strength of 2,000,000 psi in 0.5- to 1-mm-diam fibers. His objection is, therefore, to an *intrinsic* statistical flaw concept, but not to *treatment* surface damage. The high strengths observed with very thin fibers would then simply be attributed to their pliability, which prevented the occurrence of damage when rubbing one against another during handling. Similar observations were made by Bartenev and Izmailova (Ref. 60), who, by a special technique, managed to obtain flawless fibers. They report no dependence of strength on either length or diameter, and an extremely small scatter of strength data.

If an attempt should be made to draw conclusions from all of the investigations that have been made of glass fibers, the following points might be noted:

(1) Although they are certainly the cause of a size effect, flaws are more the result of damage (mechanical, hygrometric, or chemical) during aging than of an intrinsic flaw population. As a result, flaws are subject to arbitrary distribution functions.

(2) With the removal of surface flaws (by etching, etc.), near-theoretical strengths are approached. Also, size dependence and high scatter are almost eliminated. This suggests that, if the strain-energy size effect is operative (which is doubtful in view of the remoteness of fiber size from the transition size of 1 mm observed by use of the indenter technique), it is very limited in magnitude. This, however, does not contradict the proposed theory because the strain-energy size effect takes place only during the stable-propagation stage, which does not exist in the absence of notches. Generally, the two effects coexist: flaw statistics affects initiation and strain energy affects propagation. Neither exists in the absence of flaws. In the presence of a single flaw (or a dominating notch), only the strain-energy effect exists.

(3) The fact that extrapolations of test strengths to zero specimen size yield excessive theoretical strengths suggests either that the functions are too steep or that perhaps another mechanism is operative to add size sensitivity. Such a mechanism could be the strain-energy size effect.

(4) The common observation that scatter of strength data is too small, even where no surface improvement is made, may also point to the coexistence of the strain-energy mechanism, which favors the decrease of scatter data with size.

(5) There seems to be no doubt that other causes (probably the result of different properties reached during the manufacture of different sizes of fibers) contribute to the inconsistency of observations.

According to the proposed theory, a glass specimen is as brittle as it seems because, in most cases, it is too large to permit the development of stable cracking. It is theorized, however, that glass has in common with most other materials an energy-dissipating mechanism that is manifested only in very small specimens (this would make the energy-absorption curve concave upward). Marsh (Refs. 61 and 62) presents convincing arguments

to the effect that glass has a capacity for plastic deformation at stresses appreciably below the theoretical cohesive strength. His evidence includes the following:

(1) Plastic furrows produced by a hard point on a glass surface.

(2) Diamond hardness impressions.

(3) Residual stresses observed at the root of cracks after removal of load (Dalladay and Twyman, Ref. 63).

(4) Gross plastic flow found under high isotropic pressures (Bridgman and Simon, Ref. 64).

It is perhaps significant that, in all cases where plasticity was observed in glass, it was either in very small bodies or the observations were made when the stress was concentrated in very small volumes. An exception is the case of Bridgman and Simon; there the plasticity could have been gross because premature cracking was prevented by the confining pressure. In all other instances, the smallness of the stressed material apparently prevented the catastrophic generation of cracks, and this is in accord with the proposed theory. With the premature, catastrophic cracking arrested, the stresses could be raised sufficiently to accomplish the very striking plastic furrows and indentations demonstrated by Marsh (see Refs. 61 and 62). It is highly unlikely that similar plastic strains can be obtained if a *large* body of glass is subjected to stress. Brittle fracture would quickly put an end to such an attempt. It is also possible that a very careful microscopic examination would reveal the opening of stable cracks in the deformed zones of the furrows and indentations reported by Marsh.

It is instructive to note that Roesler (see Ref. 53), in extrapolating his equation for the stability condition under an indenter to very small contact areas, found critical stresses of 10^5 kg cm^{-2}. He concluded: "Apparently under tools fine enough to produce such extremely small dents, the fracture might cease to be brittle, since the stresses demanded by the energy balance condition of a brittle crack would suffice to destroy the cohesion of the material. It is perhaps possible that this speculation gives the explanation of the phenomenon of superhardness and micro-plasticity discovered by Smekal and coworkers" (Refs. 65 and 66). Although this observation is very true, it is incomplete in that it does not provide for the arrest of the propagating crack. It is the smallness of the stressed body that provides for this arrest, and thus permits the manifestation of Smekal's "micro-plasticity."

Table 2. Fracture surface energies at 25°C

Material	$\gamma_{calc} \times 10^{-3}$ ergs/cm^2	$\gamma_{obs} \times 10^{-3}$ ergs/cm^2	Reference
Steel	1.0	1000	68
Glass	1.7	0.55	69
PMMA[a]	0.5	200	67
Concrete	0.8[b]	10	70, 71

[a]Polymethylmethacrylate.

[b]Based on a 1:3 volume ratio of cement to quartzite aggregates.

Glassy polymers behave macroscopically in a manner very similar to glass; i.e., in an apparently brittle manner. However, it has long been known that these polymers have much more highly developed energy-dissipating mechanisms than glass. Berry (Ref. 67) presents the information shown in Table 2 for discrepancies in surface energies between calculated and observed values (the author has included the corresponding values for concrete). The discrepancy in polymethylmethacrylate (PMMA) is almost the same as that of steel, and the cause is similar. In both steel and PMMA, a great amount of energy is dissipated ahead of the advancing crack. In fact, Table 2 shows that the pure surface tension of PMMA is only 0.25% of the energy spent irreversibly. Independent studies of the fractured surfaces of PMMA, using such techniques as interference microscopy, reveal that the surface was affected to a depth of approximately the wavelength of visible light. The mechanism is complex; for the present purpose, it suffices to state that crazes are formed to that depth, dissipating a large amount of energy in the process.

Therefore, it is to be anticipated that PMMA will have a much more pronounced stage of stable crack propagation than does glass. Indeed, such were the conclusions reported by Berry (Ref. 72) based upon both velocity measurements (0.1 cm s^{-1} for the slow stage vs 10^4 cm s^{-1} for the catastrophic stage) and fracture-surface examination. The latter is very revealing in that it enables the accurate determination of the location of the velocity transition. The slow region has a very rough and irregular surface that is delineated by a sharp boundary; beyond this boundary, in the fast region, the surface is initially smooth and highly reflective, and then becomes duller, losing reflectivity. Thus, the fast region actually represents subregions of moderate and high velocities.

Berry (Ref. 73) has shown that the size of the initial crack affects the relative portions of the various types of

surface in such a way that, as the size of the initial crack decreases, the portions of the lower velocities decrease and the initial acceleration of the crack increases. This is in complete agreement with the theory presented herein because a specimen with a small initial crack is equivalent to a "large" specimen (see Sections III-B and XI).

The surface appearance also helps to differentiate PMMA behavior from that of glass. Smekal (Ref. 74), Shand (Ref. 75), and others have shown a similar appearance in glass except for the absence of the rough initial portion, which should not exist in a material that is "large" in the sense of the proposed theory. In his investigation, Shand calls the stage of smooth surface "slow propagation," but it can now be appreciated that it was actually a case of "load-relaxation stability" and not of "energy-dissipation stability" (see Section III). This must have been the result of his loading arrangement, which entailed load relaxation with the growth of the crack. No "energy-dissipation stability" could be expected in glass specimens of 0.2 in. diam, as were those reported by Shand.

Berry (Refs. 76 and 77) reported two experiments in which he determined γ for PMMA and for polystyrene by tensioning specimens in which natural cracks of varying depth had been previously made. He used specimens varying in cross section from 0.42 × 0.063 to 0.98 × 0.19 in. for PMMA, and from 0.42 × 0.2 to 1.42 × 0.2 in. for polystyrene; in neither case did he observe any size effect. In view of the high capacity of these materials to dissipate energy, these results are not surprising, the specimens Berry used being definitely too small and too far from their transition size.

It is known that some glassy polymers that are brittle in tension are quite ductile in compression. This is an indication that these materials actually have some capacity to yield; in the case of tensile loading, however, this stabilizing factor is not enough to control the very unstable crack growth. In compression, as has been shown, the growth is much more stable; therefore, this yielding is sufficient to control the cracks. The prevention of premature cracking thus permits the manifestation of plasticity in compression.

The present author is unaware of other works demonstrating size effects in glassy polymers. In view of the pronounced capacity of these materials to dissipate energy, however, they undoubtedly have a transition size much larger than that of inorganic glass.

X. The Three Conditions of Fracture

Because Griffith used glass for his classical experiments, his specimens were "large" in the sense of the present theory. The instability condition was assured in advance, and he met instability together with initiation. Consequently, he did not have an opportunity to learn of the more general case in which stability follows the initiation of a crack. Also, he did not realize that γ (the surface tension that he obtained for glass) was only one of two material constants required for the complete description of the material resistance to fracture. As a result of the work of Marsh (see Refs. 61 and 62), however, investigators are now in a position to realize that glass should also have a second, higher constant, which represents its ultimate resistance to fracture. This constant is usually denoted G_c for the less brittle materials.

Because Irwin and Orowan developed their theories from tests conducted with metals ("small" in the sense of the present theory), the instability condition was not satisfied *a priori*. When a crack initiated, therefore, it was stable, and they had no means of knowing of its existence. They continued to load until instability was reached, at which point they measured G_c. They did not consider the initiation of cracking to be significant, nor were they aware that the material had gone through *two* critical phases. The first phase is the initiation of the first crack, and is governed by a constant with dimensions equal to (but value much lower than) G_c. In glass, this initiation constant is γ. In metals, no experimental evidence exists to indicate what value this constant has in relation to γ. Therefore, a new value γ' is here defined as the limiting value that (½) G_c (as is known, under these conditions, $G = 2\gamma$) will approach as the specimen size increases to infinity. Future studies will reveal how close γ' is to γ, but at present it will suffice to state that γ' is the lower fracture constant.

Thus, in the general case, every material has two constants that fully describe its resistance to fracture. The lower constant γ' has already been defined, and it is also known that, for glass, $\gamma' = \gamma$. For consistency, the upper constant is here defined as the limiting value that G_c will approach with the decrease of the specimen size to zero; this constant is denoted by G_i. Strictly speaking, reducing the size to zero seems an absurdity because the whole concept of the energy balance is based upon the concentration of energy ahead of the crack—and without material there is no crack and no concentration. Consequently, with the size approaching zero, the failure criterion must change from one of energy to one of

stress, with the limiting value being the yield stress σ_y. For practical purposes, however, there is no need to approach zero because G_c attains a more or less constant value at finite sizes. The criterion still being energetic at these sizes, G_i will be used as defined above, with the realization that

$$G_i \rightarrow \frac{\pi c_i \sigma_y^2}{E} \tag{12}$$

Because both G_i and σ_y are material constants, c_i must also be a constant—one whose value is hard to estimate at this stage, but which may actually be the entire critical size of the specimen.

That σ_c approaches σ_y with decreasing specimen size is shown for steel by Lubahn's curves (see Fig. 3); for glass, this relationship is demonstrated by the fact that theoretical strengths are approached with decreasing fiber diameters. In view of Marsh's results (see Refs. 61 and 62), it is to be anticipated that yield will take place before these theoretical strengths are reached.

The two new material constants γ' and G_i control the two critical events of fracture, crack initiation and ideal instability, respectively. The relevant conditions, therefore, are as follows:

(1) Crack-initiation condition:

$$\sigma^2 c = \frac{2E\gamma'}{\pi} \tag{13}$$

(2) Ideal-instability condition:

$$\sigma^2 c = \frac{EG_i}{\pi} \tag{14}$$

or

$$\sigma = \sigma_y \tag{15}$$

where both γ' and G_i are, by definition, independent of size. Whereas γ' always determines the initiation of cracking, and instability only rarely (for very large elements), G_i rarely determines instability (for very small elements only). In all practical cases, G_c (the size-dependent value) controls instability. Thus, it is very important to make a study of the size dependency of G_c.

The instability condition in the general case, therefore, is not (2), above, but the following:

(3) Instability condition:

$$\sigma^2 c = \frac{EG_c}{\pi} \tag{16}$$

Wherever it is stated in this report that the instability condition had been assured, the meaning is that the specimen was of a sufficient size for G_c to be reduced to $2\gamma'$. In such cases, initiation and instability coincided.[7]

For practical purposes, it is most important to know G_c and how it varies with size (its size dependence is of a higher degree than that of σ or c because it is proportional to $\sigma^2 c$). It is also important to know γ' and G_i for the following reasons:

(1) So far in use only with glass (in the form of γ), γ' should be the true design criterion for very large-scale steel members. For ordinary-sized members, it is also important because it determines the event of crack initiation.

(2) So far not in use at all, G_i should be the true design criterion for small elements of glasslike materials. It also determines the extent of plastic strains that can be obtained with such materials.

For some values of the above constants, Table 2 should again be consulted. It may now be realized that the values appearing in the column γ_{obs} are actually (½) G_c; as such, they have meaning only in relation to specific sizes. The only absolute value in Table 2 is γ_{obs} for glass. Because it is obvious that the glass specimens were "large," γ_{obs} is actually γ'. For G_i of glass, Marsh's estimate is that what he termed the "fracture energy" is 50 times greater than the surface energy. However, it is certain that Marsh did not go to the limit because his results were obtained from finite-size elements. The true value (theoretically pertaining to zero size) should, therefore, be more than 50 times greater than the surface energy, and G_i should be more than 100 times greater. More explicitly, $G_i > 55{,}000$ ergs cm^{-2}. (Table 2 shows γ_{obs} to be somewhat smaller than γ_{calc}; this, in principle, is impossible.) For the remaining three materials listed in the table, no conclusion can be drawn with regard to

[7]In Griffith's experiments, because initiation and instability coincided, the term "Griffith's condition" is used rather loosely for both events. To avoid ambiguity, this term is not used in this report, and each event is given its own descriptive name (see Section I).

either γ' or G_i values except to state that the former are less and the latter more than the γ_{obs} values.

One may argue that, for metals whose transition size is very large (see Section XI,) the elements on which fracture-toughness tests are conducted are "small" by comparison, and that, as a result, the observed G_c is approximately equal to G_i. Unfortunately, however, all G_c determinations start from a notch. The notch imparts brittleness because of its triaxiality of stress, and this is equivalent to an increase in size. Thus, although the specimen may be of small dimensions, it is "large" in the sense of the present theory, the observed G_c being well below G_i. A check as to how close one approaches G_i is the value of σ_c at instability, as, according to Eq. (12), $\sigma_c = \sigma_y$ when $G_c = G_i$. Lubahn has approached this state (see Fig. 3).

Carman, Armiento, and Markus (Ref. 78) measured fracture toughness for a large variety of aluminum alloys. In all cases, σ_c was less than σ_y; in cases of purer material, however, the ratio σ_c/σ_y was appreciably higher than in cases of commercial alloys. This indicates that the specimens, which were 20 in. wide, were not "small," and that the G_c values observed were well below G_i. For the purer aluminum–i.e., the more ductile, and thus the equivalent of "smaller" specimens–the results were nearer to G_i.

The drawback of the fracture-toughness test is that it stems from the brittleness imparted by the notch; yet it is safe because it results in an underestimate of G_c. However, strictly speaking, the G_c value thus determined should be related to a much larger specimen when the study of the size dependence of G_c is made.

XI. The Transition Size

All curves representing the size dependence of strength and semiductility (i.e., ductility accompanying cracking) have the characteristic shape of a reversed sigmoid, as shown schematically in Fig. 9. In most cases, only a portion of the curve has been explored (see Figs. 3, 6, and 7), but it is obvious that the curve flattens out on both sides of its steep portion. The upper limit of the strength curve is necessary to represent the theoretical strength or, what is more likely, the yield strength. Its lower limit is necessary to represent the initiation stress to which the strength is reduced with very large elements. This reasoning would require the left-hand segment of both the strength and ductility curves to be horizontal. It would also require the right-hand segment of the strength curve to be slightly decreasing, and that of the ductility curve to be slightly

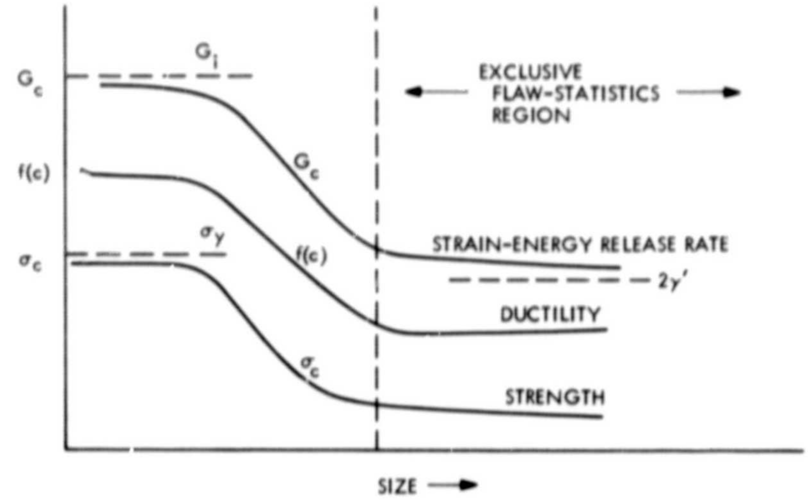

Fig. 9. Size dependence of strength, ductility, and strain-energy release rate of a material (schematic)

increasing, to represent the size effect upon initiation according to the flaw-statistics theory. At least with regard to strength, experiments confirm this prediction.

Figure 9 also shows the effect on G_c with the limiting values of G_i and γ'. The difference between the two plateaus in each of the three curves is equal to the strength, the ductility, and the G_c value that would be gained during the stable propagation of a crack in an extremely small specimen of the material. If Fig. 2 represents such a small specimen, so that σ_c, c_c, and G_c become $\sigma_i(=\sigma_y)$, c_i, and G_i, respectively (the suffix i symbolizing *ideal* instability), then the differences $\sigma_i - \sigma_1, f(c_i) - f(c_1)$, and $G_i - G_1$ are, respectively, these gains in strength, ductility, and G_c.

In principle, all materials exhibit the above behavior. They vary, however, in both the magnitude of the transition and in its position. Hardly any data are available on transition magnitude, and very little on position. However, the data presented in this report can serve as a guideline for determining the position of the transition size of various materials along a certain size scale (Fig. 10). It must be realized, however, that a more accurate determination of these transitions is impossible because size is not the only factor affecting the transition (in none of the referenced works is the size effect isolated).

Thus, it is known that the brittleness–ductility transition is affected, perhaps more than by size, by temperature, strain rate, and triaxiality of stress. In all of the examples cited in this report, all four factors acted simultaneously. In particular, the effect of stress triaxiality is confusing in the case of ductile metals because such materials are highly sensitive to this factor. In the work of

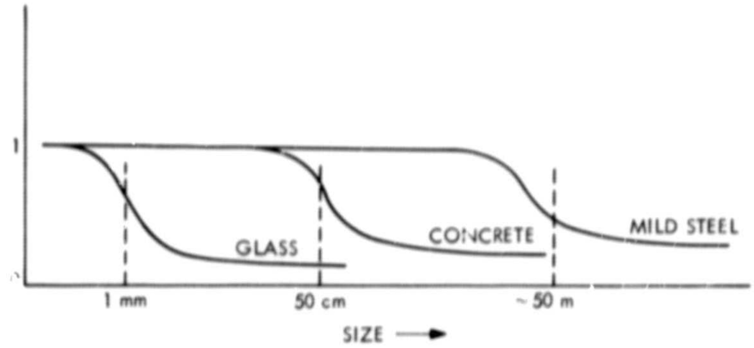

Fig. 10. Transition size for some typical materials

Lubahn (see Ref. 24) with mild steel, for example, the transition was made with specimens 22 in. long; however, these specimens were very sharply notched—a provision that enormously decreased the ductility of the steel. It is certain that no size effect would have been encountered with specimens of this size had it not been for the notches.

This is further exemplified by the other work of Lubahn (see Ref. 23) with a less ductile metal (nickel-molybdenum-vanadium steel), as shown in Fig. 3. Unnotched beams showed no size sensitivity; with increasing notch sharpness, the size sensitivity became more and more pronounced. It thus seems as if the notches shifted the curves to the left by amounts that increased with the sharpness of the notch. Therefore, if one attempts to eliminate the triaxiality effect, these curves must be shifted back to the right by equal amounts; i.e., the 0.001-in.-radius curve will be shifted the most, and the others will be shifted by smaller amounts, depending upon their sharpness. It is easy to see that the resulting curve will resemble those shown in Figs. 9 and 10. The amount of the shift depends on the sensitivity of the material to triaxiality—a problem that is still to be studied. The position of the mild-steel curve in Fig. 10 is, at this stage, no more than a guess. To determine its true position, tests will have to be conducted with very large and unnotched specimens.

A distinction should be made between triaxiality created by the stress concentration in front of the crack, which is present in all cases, and that which is imposed upon the specimen by its shape, its dimensions, and by the state of the applied stress. The effect of triaxiality upon the transition, discussed in the preceding paragraph, concerned the geometry of the specimen; e.g., that of a notch or the thickness of a plate. As far as the latter is concerned, it is customary to denote the G_c value for a plane-strain case by G_{Ic} and to regard it as a different quantity; actually, however, the two are one quantity, which is sensitive to the triaxiality of the stress.

If a study is made of the separate effects of size, triaxiality, temperature, and strain rate on strength, ductility, and G_c value, the combined effect of all these factors on any of the properties is obtained by superposition. Alternatively, in any one study, if more than one factor is involved (as were size and triaxiality in Lubahn's study), then factors can be eliminated by proper shifting of the curves, leaving the transition curve for any desired single factor.

It should be noted that a notch actually has a dual function. It causes stress triaxiality, as explained, but it also concentrates the energy in a small volume of the test specimen so that the specimen is made "smaller." An alternative explanation of this second function follows. With a notch, as opposed to a nearly smooth surface, the growth begins under a lower overall stress (since $\sigma \propto 1/c^{1/2}$). Thus, the entire specimen is almost load-free, and only the region ahead of the notch is highly loaded; therefore, the *effective* specimen is made "smaller." With very shallow notches (simulating the state of a smooth surface), the specimen is "large" because σ is high and more evenly distributed within the material (see also Section III-B).

The two effects of a notch are thus contradictory—triaxiality enhances instability, and strain energy prevents it. However, the triaxiality effect is of little importance in the case of materials that are already brittle, such as glass and concrete, whereas the strain-energy effect is of little importance in "small" materials, such as mild steel. Therefore, for all practical purposes, a notch is an embrittling factor for mild steel and a ductilizing factor for glass. Notches thus tend to converge the transition curves of the various materials toward the center of the size scale. Because some notches always exist in all practical cases, the true spectrum of transition sizes is generally wider than might be judged from experiments.

XII. Strain-Energy Effects During Various Stages of Fracture

It may be useful to summarize the effects of the strain-energy content and size at the various stages of fracture. These are as follows:

(1) The effects on the initiation of cracking are mainly a result of the statistically distributed flaws.

(2) The effect upon total fracture includes item (1), above, plus the effects during stable crack propagation.

(3) During stable propagation ("energy-dissipation stability"), the effects are as follows:

(a) The energy content determines the amount of stress relaxation with every increment of growth of the crack. A rigid system (small size) contributes to stress-relaxation stability; because the system is stable anyway, however, this has no effect upon the critical values of stress and strain.

(b) The energy content determines whether dynamic effects will develop with fluctuations of the restraining force $\partial W/\partial c$. High energy content is conducive to such effects. With low energy content, the driving force does not exceed the restraining force, and the dynamic situation does not develop.

(4) During stable propagation, the size (but not the energy content) determines the dynamic response of the system if a dynamic situation had developed according to case (3b), above. The larger the body, the lower its natural frequencies, and hence the earlier will be the onset of instability. This affects the critical values of stress, strain, crack length, dissipated energy, and G_c.

(5) During unstable propagation, the strain-energy content determines the amount of stress relaxation caused by exhaustion of energy, as described for the stable stage—case (3a), above. However, because it occurs in this case during instability (i.e., beyond the critical point or maximum stress), the stress relaxation causes either a pseudostability (see Fig. 4b) or an arrest in accordance with Eq. (8). This, of course, does not affect any of the critical values.

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